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Monetary Policy and Heterogeneity: An Analytical Framework

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Monetary Policy and Heterogeneity: An Analytical Framework[I]

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Abstract

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Keywords: heterogeneity; inequality; risk; liquidity; tractable HANK; optimal monetary policy; forward guidance puzzle; interest rate rules; determinacy; multipliers.

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1 Introduction

A spectre is haunting Macroeconomics—the spectre of Heterogeneity. Some of the world's leading policymakers have been asking for research on it, and its other name, "Inequality", in connection with stabilization, monetary and fiscal policies. For until recently, research on these two topics has been, with few exceptions, largely disconnected. Yet a burgeoning field emerged as a true synthesis of these two lodes: heterogeneous-agent (HA) and New Keynesian (NK), leading to HANK.[1]

The vast majority of contributions consists of quantitative models, involving heavy machinery for their resolution, the price to pay to achieve the realism conferred by matching the micro data.[2] Yet given that much of the post-2008 bad press of existing DSGE models refers to their being too complex and somewhat black-box, it seems important to build simple tractable representations of these models to gain analytical insights into their underlying mechanisms and make their policy conclusions sharper and easier to communicate. The two, quantitative and analytical approaches are thus strongly complementary and reinforce each other.

With this paper, I wish to suggest a tractable HANK model, **THANK**, focusing on the distinction between cyclical inequality and cyclical risk and their importance for understanding the workings of HANK models. To achieve this, I first outline THANK as a projection of rich-heterogeneity quantitative HANK along several key dimensions. I then exploit the tractability to elicit in closed form the distinct roles of cyclical inequality and risk for the model's dynamic properties: determinacy of interest rate rules, curing the forward guidance (FG) puzzle, amplification and fiscal multipliers; and optimal monetary policy.

THANK is a three-equation model isomorphic to the textbook representative-agent (RANK) model, which it nests; yet it captures several dimensions that the recent quantitative literature finds important for the study of macro fluctuations with heterogeneity. First, it features a key aggregate-demand (AD) amplification, the "New Keynesian cross" present in any HANK model where some households are constrained hand-to-mouth. Heterogeneity shapes aggregate outcomes through *cyclical inequality*: how the distribution of income between constrained and unconstrained changes over the cycle, e.g. who suffers more in recessions. This originates in the TANK model in Bilbiie (2008, 2020), generalizes to the subsequent rich-heterogeneity literature (Auclert (2019)), and has recently been investigated empirically (Patterson (2023)).

Second, my analytical model incorporates precautionary, self-insurance saving, and the distinction between liquid and illiquid assets, staples of quantitative HANK models, e.g. Kaplan et al (2018). Third, it also encompasses risk; I propose a decomposition of cyclical income risk into one part due to cyclical inequality and one related to (conditional) skewness: cyclical variations in the likelihood of ending up in the bad state.[3] I show that this distinction is key for some of the theoretical model properties. Finally, the version with liquidity allows an analytical solution for the key "intertemporal MPCs" (iMPCs) that Auclert, Rognlie, and Straub (2023) introduced in a quantitative HANK, extended here to study the novel interaction with cyclical inequality. To the best of my

[1] The abbreviation is due to Kaplan, Moll, and Violante (2018); the opening sentence is a paraphrase of Marx and Engels.

[2] Overwhelming evidence was long available for the failure of an aggregate Euler equation, for a high fraction of households having zero net worth and a high marginal propensity to consume MPC, "hand-to-mouth". Important work clarified the link with liquidity constraints: some "wealthy" households behave as hand-to-mouth if wealth is illiquid (Kaplan and Violante (2014)), perhaps because it consists of a mortgaged house (Cloyne, Ferreira, and Surico (2015)).

[3] Relatedly, the model delivers key stylized statistical properties of idiosyncratic income emphasized by a large empirical literature: autocorrelation, cyclical variance, negative skewness, and leptokurtosis.

knowledge, to this date, THANK is the only tractable framework that simultaneously captures *all* of these important features of the rich micro-heterogeneity models and allows gaining insights into their mechanisms. This burgeoning literature is reviewed in detail in Appendix A, including my own past work.

Armed with this representation, the paper studies analytically the role of cyclical inequality for the model's dynamic properties—in particular in relation to cyclical risk. Aggregate dynamics depend chiefly on the cyclicality of inequality, that is on the constrained agents' income elasticity to aggregate income χ. AD-amplification occurs with countercyclical inequality, when $\chi > 1$: a demand increase leads to a disproportionate increase in constrained agents' income and a further demand expansion; intertemporally, this also delivers *compounding* in the aggregate Euler equation. Conversely, procyclical inequality $\chi < 1$ leads to AD-dampening and Euler-equation *discounting*.

Understanding determinacy properties when inequality is cyclical is crucial for being able to even solve quantitative models, and for policy: what rules or institutions can anchor expectations when distributional mechanisms are at play. The determinacy properties of Taylor rules reflect the above intuition, materializing into a modified Taylor principle. When inequality is countercyclical, the central bank needs to be possibly much more aggressive than the "Taylor principle" (increasing nominal interest more than inflation) to rule out indeterminacy. Whereas in the discounting, procyclical-inequality case, the Taylor principle is sufficient but not necessary: for a large region there is determinacy even under a peg, undoing the Sargent-Wallace result.

A Catch-22 for HANK models thus emerges: AD-amplification relative to RANK and multipliers—that much of the literature uses HANK models for—require countercyclical inequality $\chi > 1$. Yet ruling out the forward guidance puzzle—that the later an interest rate cut takes place the larger its effect today (Del Negro, Giannoni, and Patterson, 2023)—requires the opposite: procyclical inequality $\chi < 1$; evidently, $\chi > 1$ aggravates the puzzle.

The paper then proposes solutions to the Catch-22, consisting of both model ingredients and alternative policy rules. A key contribution is emphasizing the different role for transmission and the decomposition of two cyclical channels: income inequality and, as previously emphasized by others and reviewed below, risk. My model includes a novel formalization of the latter that can also give rise to dampening/amplification, but through a distinct mechanism: if uninsurable risk increases in expansions, precautionary saving leads agents to cut back demand; if it falls in expansions (is countercyclical), agents dis-save and augment their demand. The Catch-22 is resolved in a model where both inequality and risk are cyclical, as long as their cyclicalities have the opposite sign: so that if e.g. risk is countercyclical and yields AD amplification, inequality is procyclical enough to cure the FG puzzle. Figure 1 zooms in on the last two U.S. recessions to show that inequality in disposable income (post taxes and transfers) has been *procyclical*—although inequality in factor income (labor earnings plus capital income) was strongly countercyclical, as was ex-ante income risk.[4]

[4] The reason is that transfers are highly countercyclical. Below, I review existing evidence on the cyclicality of inequality and risk, e.g. Heathcote et al (2023) and Guevenen et al (2014). Other modelling choices circumvent the Catch-22 altogether (e.g. Auclert et al assume proportional incomes i.e. acyclical inequality and risk), or augment the model with information frictions that impart a separate source of Euler discounting.

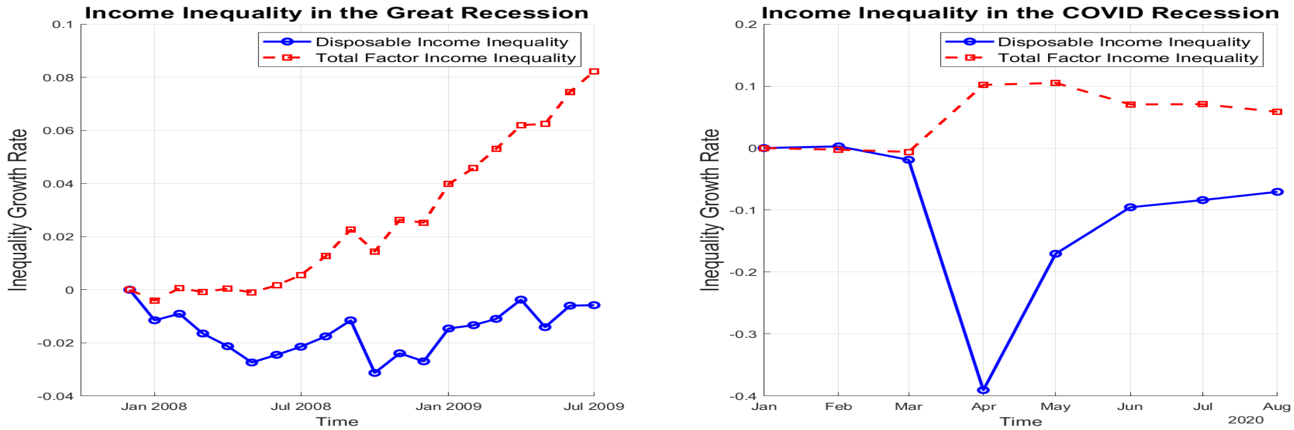

Fig. 1: Income inequality (top/bottom 50%) in the last two recessions; realtimeinequality.org data (Blanchet et al, 2023)

A different class of solutions consists of alternative policy rules that solve the Catch-22 regardless of how countercyclical inequality and risk are: they preserve amplification while ruling out the FG puzzle by determining the price level. I show that a Wicksellian price-level targeting rule achieves this in THANK. A debt quantity rule (Hagedorn (2020) and Hagedorn et al (2018)) has the same virtue. In my model's version with liquidity, I prove analytically both this and a version of Auclert et al's numerical determinacy criterion based on iMPCs but extended to embed the cyclical inequality channel; for the latter, I thus derive novel analytical expressions for the iMPCs.

The analysis of optimal monetary policy further underscores the importance of distinguishing the different roles of inequality and risk. In the benchmark I study, whereby fiscal policy is in charge of fixing long-run inequality distortions and there is no liquidity, risk is irrelevant for optimal policy, while cyclical inequality is instead a key input. Complementary to quantitative studies that address phenomenal technical challenges, I calculate optimal policy analytically in THANK, approximating aggregate welfare to second-order to derive a quadratic objective function for the central bank. This encompasses a novel inequality motive, implying optimally tolerating more inflation volatility when more households are constrained. While inequality is of the essence for optimal policy, risk is not—insofar as the policymaker shares society's first-best (perfect-insurance) objective. Risk does matter for implementation: with countercyclical inequality and idiosyncratic risk, the interest-rate rule that implements optimal discretionary policy may entail cutting real rates, when in RANK it would imply increasing them. Furthermore, optimal policy under commitment ensures determinacy regardless of heterogeneity and inequality-cyclicality and, while affected by similar inequality considerations, amounts to a form of price-level targeting.

2 THANK: An Analytical HANK Model

This section outlines THANK, an analytical model that captures several key channels of complex HANK. While related to several studies reviewed above, the exact model is to the best of my knowledge novel to this paper and its companion Bilbiie (2020), which uses a special case (with this paper as its reference for the full model), focusing on AD amplification of monetary and fiscal policies through a "New Keynesian Cross" and on using it as a one-channel approximation to richer HANK.

A unit mass of households $j \in [0, 1]$ discount the future at rate β, derive utility from consumption C_t^j and dis-utility from labor supply N_t^j, and have access to government-issued riskless bond with nominal return $i_t > 0$. Households participate infrequently in financial markets and freely adjust their portfolio when they do. When they do not, they receive only the payoff from previously accumulated bonds. Denote these two states, for ease and anticipating what equilibrium we will focus on, S from "savers" for participants and H from "hand-to-mouth" for non-participants.

The exogenous change of state follows a Markov chain: the probability to *stay* type S and H is respectively s and h, with transition probabilities $1-s$ and $1-h$; later on, I assume that s is a function of aggregate activity. I focus on stationary equilibria whereby the mass of H is the standard unconditional probability λ, and $1-\lambda$ is the mass of S:

$$\lambda = \frac{1-s}{2-s-h}, \tag{1}$$

that is the ergodic distribution in the idiosyncratic dimension around which we approximate the model with respect to aggregate shocks. TANK is nested with permanent idiosyncratic shocks $s = h = 1$ and λ fixed at its initial free-parameter value; the other non-oscillating extreme is the *iid* case $s = 1-h = 1-\lambda$: being S or H tomorrow is independent of today's state.

A particularly useful benchmark that I refer to as **oscillating THANK** abstracts from risk and focuses on *inequality only*. There is deterministic switching every period, so there is no risk; but agents oscillate between the two states, across which income inequality is still cyclical. This is nested in my model when $s = h = 0$ with agents alternating states and $\lambda = \frac{1}{2}$.[5]

Key assumptions on the asset market structure simplify the equilibrium and afford an analytical solution; the precise combination of assumptions is novel, although subsets have been used by existing literature, notably the seminal monetary-theory contributions of Lucas (1990) and Shi (1997).[6] First, households belong to a family whose utilitarian equally-weighted intertemporal welfare its head maximizes facing limits to risk sharing. Households can be in two states or "islands", with all participants on island S and all non-participants on H. The family head can transfer all resources across households *within* the island, but can only transfer some resources *between* islands.

In face of idiosyncratic risk there is thus full insurance within type, after idiosyncratic uncertainty is revealed, but limited insurance across types. At the beginning of the period, the family head pools resources within the island. The aggregate shocks are revealed and the family head determines the consumption/saving choice for each household in each island. Then households learn their next-period participation status and must move to the corresponding island, taking bonds with them. There are no transfers *after* the idiosyncratic shock is revealed, which is taken as a constraint for the consumption/saving choice. The model thus incorporates a specific notion of *liquidity*: bonds are liquid in that they can be used to self-insure before idiosyncratic uncertainty is revealed.

The flows across islands are as follows. The total measure of households leaving island H each period who participate next period is $(1-h)\lambda$; the rest λh stay. Likewise, a measure $(1-s)(1-\lambda)$ leaves island S for H at the end of each period. Total welfare maximization implies that the family head pools resources at the beginning of the period in a given island and implements symmet-

[5]The detailed outline is in Appendix C.1 for completion. This limit case is akin to Woodford (1990), abstracting from the endogenous income distribution, essential here. It is also related to the model class in Chapter 22 of Ljungqvist and Sargent (2018), ignoring the limited commitment part; I am grateful to Kurt Mitman for suggesting the latter connection.

[6]Closer to this literature, this way of reducing heterogeneity and eliminating the wealth distribution as a state variable also extends Challe and Ragot (2011), Challe et al (2017), Heathcote and Perri (2018) and Bilbiie and Ragot (2020). NK models with two switching types were studied by Curdia and Woodford (2016) and Nistico (2016) but with different insurance structures.

ric consumption/saving choices for all households in that island. Denote by B^j_{t+1} per-capita, real beginning-of-period-$t+1$ bonds on island $j = S, H$: after the consumption-saving choice, and also after changing state and pooling. The end-of-period-t (after the consumption/saving choice but *before* agents move across islands) per capita real values are Z^j_{t+1}. We have the following relations:

$$\begin{aligned} (1-\lambda) B^S_{t+1} &= s(1-\lambda) Z^S_{t+1} + (1-h)\lambda Z^H_{t+1}; \\ \lambda B^H_{t+1} &= (1-s)(1-\lambda) Z^S_{t+1} + h\lambda Z^H_{t+1}. \end{aligned}$$

Rescaling by the population masses and using (1):

$$\begin{aligned} B^S_{t+1} &= sZ^S_{t+1} + (1-s) Z^H_{t+1}; \\ B^H_{t+1} &= (1-h) Z^S_{t+1} + hZ^H_{t+1}. \end{aligned} \tag{2}$$

The program of the family head is:

$$W\left(B^S_t, B^H_t\right) = \max_{\{C^S_t, C^H_t, Z^S_{t+1}, Z^H_{t+1}\}} \left[(1-\lambda) U\left(C^S_t\right) + \lambda U\left(C^H_t\right)\right] + \beta E_t W\left(B^S_{t+1}, B^H_{t+1}\right)$$

subject to the laws of motion for bond flows (2) and budget constraints. Y^j_t are post-tax incomes, which for S also include *dividends* D_t from holding firm shares; these can thus be regarded as completely-illiquid across islands, "immobile" assets. All households receive the real return on their respective bond holdings, with i_{t-1} nominal interest and π_t net inflation, and face positive constraints on new bonds (4).

$$C^S_t + Z^S_{t+1} = Y^S_t + \frac{1+i_{t-1}}{1+\pi_t} B^S_t,$$

$$C^H_t + Z^H_{t+1} = Y^H_t + \frac{1+i_{t-1}}{1+\pi_t} B^H_t \tag{3}$$

$$Z^S_{t+1}, Z^H_{t+1} \geq 0 \tag{4}$$

The Kuhn-Tucker conditions with complementary slackness are:

$$\begin{aligned} U'\left(C^S_t\right) &\geq \beta E_t \left\{ \frac{1+i_t}{1+\pi_{t+1}} \left[sU'\left(C^S_{t+1}\right) + (1-s) U'\left(C^H_{t+1}\right)\right]\right\} \\ \text{and } 0 &= Z^S_{t+1}\left[U'\left(C^S_t\right) - \beta E_t \left\{ \frac{1+i_t}{1+\pi_{t+1}} \left[sU'\left(C^S_{t+1}\right) + (1-s) U'\left(C^H_{t+1}\right)\right]\right\}\right]; \end{aligned} \tag{5}$$

$$\begin{aligned} U'\left(C^H_t\right) &\geq \beta E_t \left\{ \frac{1+i_t}{1+\pi_{t+1}} \left[(1-h) U'\left(C^S_{t+1}\right) + hU'\left(C^H_{t+1}\right)\right]\right\} \\ \text{and } 0 &= Z^H_{t+1}\left[U'\left(C^H_t\right) - \beta E_t \left\{ \frac{1+i_t}{1+\pi_{t+1}} \left[(1-h) U'\left(C^S_{t+1}\right) + hU'\left(C^H_{t+1}\right)\right]\right\}\right]. \end{aligned} \tag{6}$$

The key is the Euler equation (5), governing the bond-holding decision of S self-insuring against the risk of becoming H and taking into account that bonds can be used when moving to the H island.

Equation (6) determines the bond choice of agents in the H island; both bond Euler conditions are written as complementary slackness conditions. With this market structure, the Euler equations (5) and (6) are of the same form as in fully-fledged incomplete-markets model of the Bewely-Huggett-Aiyagari type. In particular, the probability $1-s$ measures the uninsurable risk to switch to a bad state next period, risk for which only bonds can be used to self-insure—thus generating a demand for bonds for "precautionary" purposes.[7]

To obtain the simple equilibrium representation, I focus on equilibria where the constraint of H always binds and (6) is in fact a strict inequality, whatever the reason: for instance, the shock is a "liquidity"/impatience shock making them want to consume more today (decreasing β in (6) to a low enough value β^H); or their average income in H is lower enough than in S, e.g. if profits are high enough; or simply because a technological constraint prevents them from accessing asset markets.

I consider two equilibria, according to whether liquidity is supplied or not: as a benchmark, the zero-liquidity limit reminiscent of Krusell, Mukoyama and Smith (2011)[8]; and then in section 5, an equilibrium with government-provided liquidity. In the former case, we assume that even though S's demand for bonds is well-defined (their constraint is not binding), the supply is zero so there are no bonds held in equilibrium. Under these assumptions the only equilibrium condition from this part of the model is the Euler equation for bonds of S, (5) holding with equality. The H's constraint binding and zero-liquidity implies that they are hand-to-mouth $C_t^H = Y_t^H$. Because transition probabilities are independent of history and there is full insurance within type, all agents who are H in a given period have the same income and consumption.

The rest of the model is purposefully kept exactly like the TANK version in Bilbiie (2008, 2020), nested here with $s=1$. The λ households who are "hand-to-mouth" H make a labor supply decision determining their income $Y_t^H = W_t N_t^H + \mathcal{T}_t^H$, where W is the real wage, N^H hours and $\mathcal{T}_t^H$ fiscal transfers to be spelled out. The remaining $1-\lambda$ agents also work, and receive *profits* from the firm shares they are assumed to hold, net of taxes $Y_t^S = W_t N_t^S + \frac{1}{1-\lambda} D_t + \mathcal{T}_t^S$; this provides a simple mapping to the total factor income inequality data in Figure 1. The hours choice delivers the standard intratemporal optimality condition for each j: $U_C^j\left(C_t^j\right) = W_t U_N^j\left(N_t^j\right)$. Defining $\sigma^{-1} \equiv U_{CC}^j C^j / U_C^j$ as risk aversion and $\varphi \equiv U_{NN}^j N^j / U_N^j$ as the inverse labor supply elasticity, and small letters log-deviations from a symmetric steady-state (to be discussed below), we have the labor supply for each j: $\varphi n_t^j = w_t - \sigma^{-1} c_t^j$. Assuming identical elasticities across agents, the same holds on aggregate $\varphi n_t = w_t - \sigma^{-1} c_t$.

The supply, firms' side is standard and outlined for completion in Online Appendix O.A.3. A notable assumption as a benchmark is the standard NK optimal sales subsidy inducing marginal cost pricing. This policy is *redistributive*: since steady-state profits are zero $D=0$, it taxes the firms' shareholders and results in the full-insurance, symmetric steady-state used here as a benchmark $C^H = C^S = C$. Loglinearizing around it, profits vary inversely with the real wage $d_t \left(\equiv \ln\left(D_t/Y\right)\right) = -w_t$, an extreme form of a general property of NK models. This series of assumptions is not necessary for the results and can be easily relaxed, but makes the algebra more transparent.

[7]An earlier working paper version modelled explicitly a share-holding decision for firms shares as being *illiquid* in an extreme form: impossible to carry to and not receiving any dividend while in the H state. While that formulation delivers an Euler equation pricing shares only with the stochastic discount factor of S agents, its implications are otherwise isomorphic to the simpler formulation adopted here.

[8]Other zero-liquidity HANK include i.a. Ravn and Sterk (2020), Werning (2015), and Broer et al (2020).

Firms' optimal pricing under Rotemberg costs implies the loglinearized Phillips curve:

$$\pi_t = \beta E_t \pi_{t+1} + \kappa c_t + u_t, \tag{7}$$

where u_t are cost-push shocks that I abstract from until studying optimal policy in Section 6. To obtain maximum tractability and closed forms, I first focus on the simplest special case:

$$\pi_t = \kappa c_t, \tag{8}$$

nested in (7) above with myopic firms ($\beta = 0$), used previously in a different context in Bilbiie (2019). Online Appendix O.A.3 microfounds this assuming that firms pay a Rotemberg cost relative to yesterday's *market average* price index, rather than to their own individual price (the latter leads to (7)). That is, firms ignore the impact of today's price choice on tomorrow's profits. While over-simplified, this nevertheless captures the key supply-side NK trade-off between inflation and real activity and allows us to isolate and focus on the essence of this paper: AD. All the results reassuringly generalize to the standard Phillips curve (7), as I show in Online Appendix O.C.

The government conducts fiscal and monetary policy. The former consists of a simple *endogenous redistribution* scheme: taxing profits at rate τ^D and rebating the proceedings lump-sum to H who thus receive $\frac{\tau^D}{\lambda} D_t$ per capita; this is key for the cyclical-inequality channel. In the version with liquidity, the government also supplies liquid nominal bonds and levies lump-sum uniform taxes/transfers on households. The central bank sets the nominal interest rate i_t.

Market clearing implies for equilibrium in the goods and labor market respectively $C_t \equiv \lambda C_t^H + (1-\lambda) C_t^S = \left(1 - \frac{\psi}{2}\pi_t^2\right) Y_t$ and $\lambda N_t^H + (1-\lambda) N_t^S = N_t$. With uniform steady-state hours $N^j = N$ by normalization and the fiscal policy assumed above inducing $C^j = C$, loglinearization around a zero-inflation steady state delivers $y_t = c_t = \lambda c_t^H + (1-\lambda) c_t^S$ and $n_t = \lambda n_t^H + (1-\lambda) n_t^S$.

2.1 Cyclical Income Risk and Inequality in THANK

A keystone to this paper's analysis is to distinguish income inequality and risk, and their cyclicality. I define *income inequality* as the income ratio in the two states $\Gamma_t \equiv Y_t^S / Y_t^H$; this is proportional to the unconditional variance of log income (and also to the Gini and entropy, see Appendix O.B.1):

$$var\left(\ln Y_t^j\right) = \lambda (1-\lambda) (\ln \Gamma_t)^2 .$$

Importantly, in the model's equilibrium inequality is cyclical: it depends on aggregate output $\Gamma(Y_t)$.

In the data and in quantitative HANK models alike, *income risk* is cyclical. Other analytical HANKs model it as either unrelated (Acharya and Dogra (2020)) or differently related (Challe et al (2017); Holm (2021); Ravn and Sterk (2020); Werning (2015)) to liquidity constraints and hand-to-mouth status. To capture a cyclical risk component that is distinct from cyclical inequality and further differentiate from the cited papers, I assume that the probability of becoming constrained depends on *current* aggregate demand $1 - s(Y_t)$.[9] If the first derivative of $1 - s(.)$ is positive

[9] In a model with endogenous unemployment risk like Ravn and Sterk or Challe et al, this happens in equilibrium through search and matching. This is also related to Werning's Section 3.4, where nevertheless it is unconditional probabilities (and population shares) that are cyclical, and depend on *next period*'s aggregate (I study that version of my model too in Appendix

$-s'(Y_t) > 0$, the probability is higher in expansions; insofar as being constrained leads on average to lower income, this makes income risk procyclical. Conversely, $-s'(Y_t) < 0$ makes risk countercyclical.

A precise definition of "income risk" is notoriously controversial. The literature often employs the conditional variance of idiosyncratic (log) income, found to be countercyclical in the data by Storesletten, Telmer, and Yaron (2004). This is easily calculated in my two-state model as:

$$var\left(\ln Y_{t+1}^S | \ln Y_t^S\right) = s(Y_t)(1 - s(Y_t))(\ln \Gamma_{t+1})^2. \tag{9}$$

The simplest oscillating THANK benchmark $s = (h =) \, 0$ is especially useful because it partials out risk: the variance (9) is *nil* with agents alternating states every period. But inequality is still arbitrarily cyclical $\Gamma_t(Y_t)$. As we will see momentarily, this is key for the model's dynamics, which we now solve for: first focusing on inequality and then enlarging the scope to broader income risk.

2.2 Cyclical Inequality and Aggregate Demand in THANK

We derive an aggregate Euler-IS equation by taking an approximation (in the aggregate-shocks dimension) around the ergodic idiosyncratic distribution with relative shares given by (1). To isolate the role of cyclical inequality, we first approximate around a *symmetric* steady-state with $\Gamma = 1$ and $C^H = C^S$. Start from the individual self-insurance Euler equation (5):

$$c_t^S = sE_t c_{t+1}^S + (1 - s) E_t c_{t+1}^H - \sigma(i_t - E_t \pi_{t+1}). \tag{10}$$

To find the aggregate(d) counterpart, we need a theory connecting the distribution c_t^j to aggregates c_t or y_t. Once idiosyncratic uncertainty is revealed and asset markets clear, this part—one of many possible examples of the distribution-aggregate feedback—is exactly the TANK model in Bilbiie (2008, 2020), for simplicity. I summarize its main implications here and refer to Online Appendix O.B for a complete derivation and to those papers for a thorough discussion. In equilibrium, individual consumptions/incomes are related to aggregate income by:

$$c_t^H = y_t^H = \chi y_t \text{ and } c_t^S = \frac{1 - \lambda\chi}{1 - \lambda} y_t \tag{11}$$

where $\chi \equiv 1 + \varphi\left(1 - \frac{\tau^D}{\lambda}\right) \lesseqgtr 1$. Equilibrium income inequality $\gamma_t \equiv \ln(\Gamma_t/\Gamma)$ is cyclical:

$$\gamma_t \equiv y_t^S - y_t^H = (1 - \chi)\frac{y_t}{1 - \lambda}. \tag{12}$$

Inequality is procyclical ($\partial\gamma/\partial y > 0$) iff $\chi < 1$ and countercyclical iff $\chi > 1$. The composite parameter χ is the model's keystone: a sufficient statistic that in this specific model depends on fiscal redistribution $1 - \tau^D/\lambda$ and labor market characteristics φ. It is important to stress that this is but one possible simple theory of the income distribution-to-aggregate two-way feedback.[10]

B). Here, to capture purely idiosyncratic (as opposed to "aggregate") variation, λ is invariant.

[10] Several subsequent models deliver different distributional implications, e.g. using *sticky wages*. See Colciago (2011), Ascari et al (2017) and Walsh (2018) in TANK; Broer et al (2020), Hagedorn et al (2018), and Auclert et al (2023) in HANK.

Distributional considerations make χ different from 1. In RANK, such considerations are absent since one agent works and receives all the profits. When aggregate income goes up, labor demand, real wages and marginal cost increase. This decreases profits, but because the same agent incurs both the labor gain and the profit loss, the redistribution of income across factors is neutral.

Take now the case with heterogeneity and *countercyclical* inequality $\chi > 1$.[11] If demand goes up, the wage goes up, H's income increases and so does their demand. Thus aggregate demand increases by more than initially, shifting labor demand and increasing the wage even further, and so on—a (New) Keynesian cross. With $\chi < 1$, inequality is *procyclical* and the AD expansion is instead smaller than the initial impulse, as H recognize that this will lead to a fall in their income.

Replacing the individual (11) in the self-insurance equation (10), we obtain the aggregate Euler-IS:

$$c_t = \delta E_t c_{t+1} - \sigma \frac{1-\lambda}{1-\lambda\chi} \left(i_t - E_t \pi_{t+1}\right), \text{ where } \delta \equiv 1 + (\chi - 1) \frac{1-s}{1-\lambda\chi}. \tag{13}$$

The AD interest elasticity is the TANK one (Bilbiie, 2008), $\sigma \frac{1-\lambda}{1-\lambda\chi}$, reflecting the described New Keynesian Cross logic.[12] I confine attention throughout to $\lambda\chi < 1$, so this stays positive; the cited paper also analyzes the "inverted Keynesian cross" region with $\lambda\chi > 1$.

The key novelty of THANK's aggregate Euler-IS equation relative to TANK is that it is characterized by *compounding* $\delta > 1$ iff inequality is countercyclical $\chi > 1$ and *discounting* $\delta < 1$ if procyclical $\chi < 1$. The stand-alone role of cyclical inequality for Euler compounding/discounting is illustrated most sharply in the *oscillating THANK* no-risk benchmark. Even in that extreme case, my model implies Euler discounting-compounding with, replacing $s = 0$ and $\lambda = \frac{1}{2}$ in (13):

$$c_t = \frac{\chi}{2-\chi} E_t c_{t+1} - \frac{\sigma}{2-\chi} \left(i_t - E_t \pi_{t+1}\right), \text{ where } \frac{\chi}{2-\chi} \lesseqgtr 1 \text{ iff } \chi \lesseqgtr 1. \tag{14}$$

This illustrates most clearly that risk is not necessary for Euler discounting/compounding: cyclical inequality is sufficient, combined with a self-insurance, liquidity motive.

In RANK and TANK, good future income news imply a one-to-one demand increase today as households (who can) substitute consumption towards the present and, with no assets, income adjusts. *Discounting* occurs when procyclical inequality meets self-insurance: When good news about future aggregate income arrive, households recognize that in some states they will be constrained and not benefit fully from it. They self-insure, increasing their consumption less than if there were no transition to another state; the saving demand increase cannot be accommodated (there is no asset), so income falls accordingly. Countercyclical inequality leads to *compounding* instead: good aggregate income news boost today's demand because they imply *less* need for self-insurance. Since future income in states where the constraint binds over-reacts to good aggregate news, households demand *less* saving. With zero savings in equilibrium, households consume more than one-to-one and income increases *more* than without transition.

[11] This occurs e.g. with fiscal redistribution of profits that is not skewed towards H, $\tau^D < \lambda$ *and* upward-sloping labor supply $\varphi > 0$. The benchmark imlicitly used by Campbell and Mankiw's (1989) is $\chi = 1$, which occurs when profits are uniformly redistributed $\tau^D = \lambda$ or labor is infinitely elastic $\varphi = 0$; income inequality is then *acyclical*. See also Bilbiie (2008, footnote 14).

[12] The direct effect scales down by $1 - \lambda$, but the indirect increases with λ at rate χ; with $\chi > 1$ the latter dominates, delivering amplification. $\lambda\chi$ is akin to an *aggregate-MPC* slope of a planned-expenditure line. As shown in Bilbiie (2020) the aggregate MPC out of *aggregate* income combines the two out-of-own-income MPCs weighed by the elasticities to the aggregate $mpc = (1-\lambda) \times (1-\beta) \times \frac{1-\lambda\chi}{1-\lambda} + \lambda \times 1 \times \chi = 1 - \beta\left(1 - \lambda\chi\right).$

These intertemporal effects are strongest in the deterministic, *oscillating THANK* case, but the same intuition applies in the general case $s > 0$ (see also Proposition 3 in the companion paper Bilbiie (2020)). The extent of self-insurance is now proportional to the level of risk $1 - s$: it vanishes in the TANK ($1 - s = 0$) limit, and increases as $1 - s$ approaches λ. Yet even the more general THANK version still focuses on cyclical inequality: while it embeds idiosyncratic risk (intimately related to binding liquidity constraints), this is by construction *acyclical*. To substantiate this and clarify the role of risk and its interaction with inequality, we now turn to the cyclicality of risk.

2.3 Cyclical Inequality and Risk

To gauge the connection between the cyclicality of inequality and risk, consider the cyclicality of the risk measure (9). A first-order approximation around steady state Γ delivers:

$$var\left(\ln Y_{t+1}^{S} | \ln Y_{t}^{S}\right) - s\left(1-s\right)\left(\ln\Gamma\right)^{2} \simeq 2\ln\Gamma\left[\underbrace{\left(1-\chi\right)\frac{s\left(1-s\right)}{1-\lambda}}_{\text{inequality}} \underbrace{-s_{Y}\left(s-\frac{1}{2}\right)\ln\Gamma}_{\text{pure risk}}\right]y_{t+1} \qquad (15)$$

The first component of (15) is due to cyclical *inequality*: when $\chi > 1$, inequality is countercyclical and so is risk because income at the bottom overreacts, increasing variance in both expansions and recessions. This "intensive" margin operates even when the second channel is absent, i.e. with constant s or symmetric distribution $s = \frac{1}{2}$. However, locally around *no-level-inequality* $\Gamma = 1$, the case we just studied, variance is acyclical: it is proportional to $\ln\Gamma$. Idiosyncratic risk itself is still cyclical to higher orders and away from steady state, but this has locally no *first-order* effect on the variance, on precautionary saving, and thus on Euler discounting-compounding. A more extreme case is TANK, $s = 1$: there is no transition and no risk in *levels*. Away from these special cases, risk is cyclical *to first order* through the inequality channel $\chi \neq 1$ only if there is level inequality $\Gamma > 1$; this is true even when the second channel is off, e.g. with *acyclical* (pure) risk $s_Y = 0$.

The second component of (15) is related to the cyclicality of *conditional skewness*, in a manner reminiscent of Mankiw (1986).[13] I derive this formally in Online Appendix O.B, but the intuition is simple: skewness is negative when $s > \frac{1}{2}$ (there is left-tail risk). When $1 - s\left(Y\right)$ is decreasing with aggregate activity $-s_Y < 0$, it becomes more likely to draw from the left tail in recessions; hence, (skewness) risk is countercyclical: upward income movements become less likely and downward *more* likely in a recession. This "extensive" margin operates even with *acyclical* inequality $\chi = 1$: risk is cyclical only through the second channel $s_Y \neq 0$, and only if there is *level* inequality $\Gamma > 1$ and skewness $s \neq \frac{1}{2}$.

While the precise decomposition (15) is model-specific, the general idea, mechanisms and their equilibrium implications transcend this model. Thus, the model nests several scenarios that are useful to distil the roles of risk and inequality through their levels and cyclicalities, and their in-

[13] Appendix O.B derives higher moments: formally, conditional skewness $\left(1-2s\right)/\sqrt{s\left(1-s\right)} < 0$ when $s > \frac{1}{2}$. Direct evidence on this conditional skewness in *levels* is not yet available; Guvenen et al (2014) and a large literature following them document countercyclical skewness in *growth* rates. Bilbiie, Primiceri and Tambalotti (2022) compute the skewness of income growth rates for a quantitative estimated version of THANK and show that the cyclicality is close to the one documented in the data by Guvenen et al.

teractions, governed by Γ, s, χ and s_Y. To reflect this, I now turn cyclical risk back on in the local dynamics, both by allowing s to depend on the cycle *and* approximating around an unequal steady state with $\Gamma > 1$. Risk is cyclical through both channels, see (15), and matters *to first order*—the loglinearized aggregate Euler-IS becomes, see Appendix B:

$$\begin{aligned} c_t &= \underbrace{\tilde{\delta} E_t c_{t+1}}_{\text{inequality}} + \underbrace{\eta c_t}_{\text{risk}} - \underbrace{\sigma \frac{1-\lambda}{1-\lambda\chi} \left(i_t - E_t \pi_{t+1}\right)}_{\text{inequality (TANK)}} \\ &= \frac{\tilde{\delta}}{1-\eta} E_t c_{t+1} - \frac{\sigma}{1-\eta} \frac{1-\lambda}{1-\lambda\chi} \left(i_t - E_t \pi_{t+1}\right) \\ &\quad \text{with } \tilde{\delta} \equiv \delta + \left(\Gamma^{1/\sigma} - 1\right)(\delta - 1)\tilde{s} = 1 + \frac{(\chi - 1)(1-\tilde{s})}{1-\lambda\chi} \\ &\quad \text{and } \eta \equiv \frac{s_Y Y}{1-s} \left(1 - \Gamma^{-1/\sigma}\right)(1-\tilde{s})\, \sigma \frac{1-\lambda}{1-\lambda\chi}, \end{aligned} \tag{16}$$

where $1 - \tilde{s} \equiv \frac{(1-s)\Gamma^{1/\sigma}}{s+(1-s)\Gamma^{1/\sigma}} > 1 - s$ is the inequality-weighted transition probability measure of risk.

This illustrates the two effects of income risk cyclicality corresponding to the decomposition (15) above. The first comes from income inequality, which around a steady-state with $\Gamma > 1$ and $s > 0$ also makes risk cyclical, captured by $\left(\Gamma^{1/\sigma} - 1\right)(\delta - 1)\tilde{s}$. When $\delta > 1$ there is an additional compounding force that increases with the inequality *level*. The second effect encapsulates "pure" (independent on cyclical inequality) cyclical risk through its key determinant: the elasticity $-s_Y Y / (1-s)$, the second term in (15) above, which determines the novel parameter η. Dampening/amplification of *both* current and future shocks occurs through η even with acyclical inequality $\delta = 1$. Procyclical risk $\eta < 0$ implies dampening and Euler discounting: interest rate cuts or good news generate an expansion today, which increases the probability of moving to the bad state and triggers precautionary saving, curtailing the expansion. Conversely, countercyclical risk generates amplification and compounding: the aggregate expansion reduces the probability of moving to the bad state and mitigates the need for insurance, magnifying the initial expansion.

This formalization of cyclical risk has similar reduced-form implications for aggregate demand to the cyclical-inequality channel, but the underlying economic mechanism is different. Furthermore, while η is related to Acharya and Dogra's (2020) different formalization of risk with CARA preferences leading to a P(seudo-)RANK, the implications are different in important respects. In my model, η captures the cyclicality of conditional *skewness*, a key element of the reviewed evidence; whereas Acharya and Dogra's PRANK relies on symmetry (normal shocks), abstracting from skewness to focus on variance. Werning's (2015) general non-linear model contains both channels but without the distinction, decomposition and discussion of their differential effects on transmission. Ravn and Sterk (2020)'s model delivers something akin to η from search and matching, but abstracts from the role of cyclical inequality. Finally, novel to my framework is the *contemporaneous* amplification ("multiplier") of current interest rate changes; this is a consequence of risk depending on *current* aggregate demand, whereas in the mentioned contributions it depends on future activity.[14]

[14] Appendix B studies an alternative where s depends on future Y_{t+1}, delivering only future amplification, thus closer to Acharya and Dogra (2020)–the within-period AD elasticity to r is then unaffected..

The pure-risk channel captured by η operates *only if* there is long-run *level* inequality $\Gamma > 1$, the literal risk of moving to a lower income level. Whereas the cyclical-inequality channel relies only on the cyclicality of income when constrained χ. Both channels capture precautionary saving: the former, through the effect of uncertainty and the third derivative of the utility function (η is proportional to prudence σ); the latter, through the effect of constraints, a separate source of concavity in the consumption function, a manifestation of the general results in Carroll and Kimball (1996).

Inspecting (16) reveals three endogenous wedges relative to RANK, corresponding to each channel; in a recent contribution, Berger, Bocola and Dovis (2023) provide a valuable data "accounting" of related wedges. Bilbiie, Primiceri and Tambalotti (2022) quantify each channel's contribution to estimated business-cycle amplification through heterogeneity and conclude that cyclical-risk (with long-run inequality level) is the most quantitatively significant. Such exercises may help disentangle the signs of $\delta - 1$ and η which, as we will see next, are crucial for the model's properties.

3 THANK Analytics: Determinacy, Puzzles, and Amplification

This section exploits tractability to conduct a pencil-and-paper analysis elucidating the dynamic properties of the model, with a particular focus on the roles of cyclical inequality and risk in shaping the determinacy properties, the FG puzzle, and conditions for amplification-multipliers.

3.1 HANK, Taylor, and Sargent-Wallace

I now solve the model under further assumptions delivering a *one-equation* representation that may be useful in different contexts where reducing dimensionality is necessary for closed-form solutions. In particular, first, the nominal rate i_t follows a Taylor rule (we study other policies momentarily):

$$i_t = \phi \pi_t. \tag{17}$$

The model is completed by adding the simple aggregate-supply, Phillips-curve specification (8); all the results carry through with the forward-looking (7) as I show in Online Appendix O.C.

With this simple RANK-isomorphic HANK we first revisit a classic determinacy result and derive a HANK Taylor principle. Replacing (8) and (17) in (13), THANK collapses to *one* equation:

$$c_t = \nu E_t c_{t+1}, \text{ where } \nu \equiv \frac{\delta + \kappa\sigma \frac{1-\lambda}{1-\lambda\chi}}{1 + \phi\kappa\sigma \frac{1-\lambda}{1-\lambda\chi}} \tag{18}$$

captures the effect of good news on AD, and the elasticity to interest rate shocks.

There are three channels shaping this key summary statistic. First, the "pure AD" effect through δ coming from cyclical inequality, operating even with fixed prices or fixed real rate $i_t = E_t \pi_{t+1}$. Second, a supply feedback *cum* intertemporal substitution: the inflationary effect (κ) of good income news triggers *ceteris paribus* a fall in the real rate and intertemporal substitution towards today, the magnitude of which depends on static amplification/dampening $\frac{1-\lambda}{1-\lambda\chi}$. Finally, all this demand amplification generates inflation and real rate movements. When policy is active $\phi > 1$, a higher real rate and a contractionary effect today ensue, the strength of which depend on cyclical-inequality. These considerations drive Proposition 1 (the case with NKPC (7) is in Online Appendix O.C.1).

Proposition 1 ***The HANK Taylor Principle***: *The HANK model under a Taylor rule (18) has a determinate, locally unique rational expectations equilibrium if and only if (as long as $\lambda\chi < 1$):*

$$\nu < 1 \Leftrightarrow \phi > \phi^* \equiv 1 + \frac{\delta - 1}{\kappa\sigma \frac{1-\lambda}{1-\lambda\chi}}.$$

The ***Taylor principle*** *$\phi > 1$ is sufficient for determinacy if and only if inequality is procyclical so $\delta \leq 1$.*

The proposition follows by recalling that the determinacy requirement is that the root ν be inside the unit circle; in the discounting case $\delta < 1$, the threshold ϕ is evidently weaker than the Taylor principle, while in the compounding case it is stronger. With countercyclical inequality and $\delta > 1$, a sunspot increase in future aggregate income generates a disproportionate increase in income in the H state and thus incentives to dis-save and a further demand boost today, making the sunspot self-fulfilling even with $\phi = 1$. The central bank needs to do more to counteract this. The opposite holds in the discounting case: the Taylor principle is sufficient for determinacy. In the *oscillating THANK* extreme, the threshold is $\phi^* = 1 + (\chi - 1) \frac{2}{\kappa\sigma}$. The Taylor threshold $\phi > 1$ reappears for either of $\chi = 1$ (acyclical inequality), $s \to 1$ (TANK), or $\kappa \to \infty$ (flexible prices). The determinacy region squeezes rapidly with countercyclical inequality because of a complementarity with idiosyncratic risk apparent from $\phi^* = 1 + \frac{(\chi-1)(1-\tilde{s})}{\kappa\sigma(1-\lambda)}$. The threshold depends on price stickiness because policy responds to inflation, but the relevant amplification goes through real demand's equilibrium response, and price stickiness modulates the relationship between the two. If instead policy responds to real activity $i_t = \pi_t + \phi_c c_t$, the determinacy threshold is $\phi_c > \frac{(\chi-1)(1-s)}{(1-\lambda)\sigma}$ and no longer depends on price stickiness because policy then acts directly on demand.

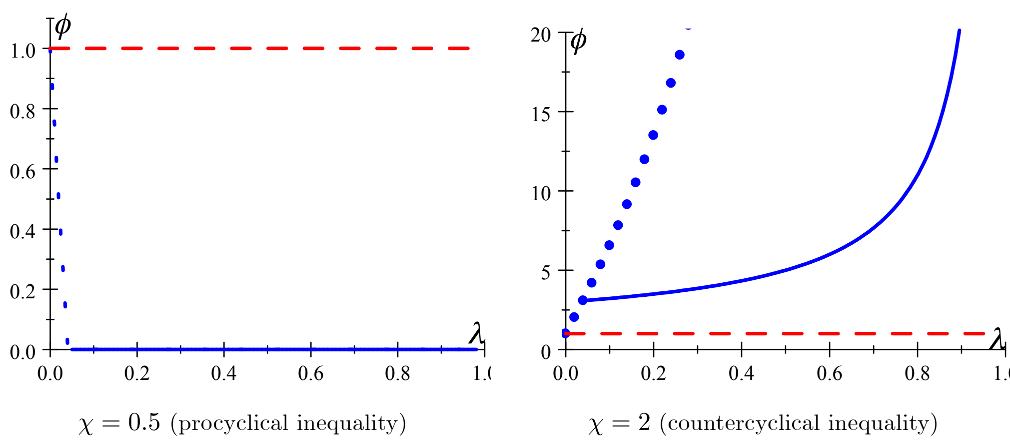

Fig. 2: Taylor threshold ϕ^* with $1 - s = 0$ (dash, TANK); 0.04 (solid); λ (dots). Note: determinacy above the curve.

Figure 2 plots the threshold ϕ^* as a function of λ (for $\lambda < \chi^{-1}$) for different $1 - s$, with procyclical inequality in the left panel and countercyclical in the right. The parametrization assumes $\kappa = 0.02$, $\sigma = 1$, and $\varphi = 1$. In the countercyclical-inequality case, the threshold increases with λ and does so at a faster rate with higher risk $1 - s$. The required response can be large: for the calibration used in Bilbiie (2020) to match Kaplan et al's HANK aggregates ($\chi = 1.48$, $\lambda = 0.37$, $1 - s = 0.04$) it is $\phi^* = 2.5$ and can approach 5 for other calibrations therein. With procyclical inequality, the Taylor principle is sufficient but not necessary for determinacy. For a large region, there is determinacy even under a peg $\phi = 0$, undoing the Sargent-Wallace result, namely if and only if $\phi^* < 0$ or:

$$\nu_0 \equiv \delta + \kappa\sigma \frac{1-\lambda}{1-\lambda\chi} < 1. \tag{19}$$

With enough discounting, the sunspot is ruled out by the economy's endogenous forces, unlike in RANK where $\nu_0 = 1 + \kappa\sigma \geq 1$; as we shall see, (19) *also* rules out the forward guidance puzzle.

Finally, a simple analogous derivation reveals the threshold when adding cyclical risk:

$$\phi^* \equiv 1 + \frac{\tilde{\delta} - 1 + \eta}{\kappa\sigma \frac{1-\lambda}{1-\lambda\chi}}, \tag{20}$$

with the different intuition discussed above for AD amplification/dampening through η.[15]

3.2 A Catch-22 for HANK: No Puzzle, No Amplification?

We are now in a position to state the Catch-22: the closed-form conditions for amplification in THANK are the opposite of those needed to solve the FG puzzle. To state this formally, we introduce two policy shocks: discretionary exogenous changes in interest rates i_t^* in the Taylor rule $i_t = \phi\pi_t + i_t^*$; and public spending: the government buys an amount of goods G_t with zero steady-state value ($G = 0$) and taxes all agents uniformly to finance it.[16] Straightforward derivation delivers the aggregate Euler-IS, starting with *cyclical inequality* only, extending (13), with $\zeta \equiv \varphi\sigma / (1 + \varphi\sigma)$:

$$c_t = \delta E_t c_{t+1} - \sigma \frac{1-\lambda}{1-\lambda\chi} (i_t - E_t \pi_{t+1}) + \zeta \left[\frac{\lambda(\chi - 1)}{1-\lambda\chi} (g_t - E_t g_{t+1}) + (\delta - 1) E_t g_{t+1} \right]. \tag{21}$$

Together with the static PC $\pi_t = \kappa c_t + \kappa\zeta g_t$ and the AR(1) $E_t g_{t+1} = \mu g_t$, this delivers Proposition 2 (see Online Appendix O.C.2 for the general case with NKPC (7)).

Proposition 2 ***A Catch-22 for HANK:*** *In THANK with cyclical inequality, there is amplification of monetary policy relative to RANK and the fiscal multiplier on consumption is positive if and only if:*

$$\chi > 1,$$

whereas the forward-guidance puzzle is ruled out ($\frac{\partial^2 c_t}{\partial(-i_{t+T}^*)\partial T} < 0$*) only if*

$$\chi < 1.$$

The *oscillating THANK* delivers the sharpest version of the Catch-22; the Euler equation is:

[15] Ravn and Sterk (2020) show that SaM delivers $\eta > 0$ endogenously, making the Taylor principle insufficient. In works *subsequent* to this paper's determinacy Proposition 1: Acharya and Dogra (2020) derived a Taylor principle and Auclert et al (2018) provided numerical simulations in a quantitative HANK on the role of *cyclical risk* for determinacy.

[16] The redistribution of the taxation financing spending is essential for the multiplier, see Bilbiie (2020) in TANK: I sidestep it assuming uniform taxation. See Bilbiie, Monacelli, and Perotti (2013) in TANK, and Oh and Reis (2012), Ferrière and Navarro (2018), Hagedorn et al (2018) and Auclert et al (2018) in quantitative HANK for multipliers with progressivity.

$$c_t = \frac{\chi}{2-\chi} E_t c_{t+1} - \frac{\sigma}{2-\chi}\left(i_t - E_t \pi_{t+1}\right) + \zeta \frac{\chi - 1}{2-\chi}\left(g_t + E_t g_{t+1}\right), \tag{22}$$

with amplification/multipliers for $\chi > 1$ but discounting and curing the FG puzzle for $\chi < 1$.

In the general case, the first part of Proposition 2 pertains to amplification, the focus of the majority of quantitative HANK. Kaplan et al (2018) show that HANK yields monetary policy amplification through indirect effects; see also Auclert (2019), Gornemann et al (2015), and Debortoli and Galí (2018). This occurs only with countercyclical inequality. The THANK *fiscal multiplier* is:

$$\frac{\partial c_t}{\partial g_t} = \frac{1}{1-\nu\mu} \frac{\zeta}{1+\phi\kappa\sigma\frac{1-\lambda}{1-\lambda\chi}} \left[\underbrace{(\chi - 1)\frac{\lambda(1-\mu) + (1-s)\mu}{1-\lambda\chi}}_{\text{TANK + HANK AD}} - \underbrace{\kappa\sigma\frac{1-\lambda}{1-\lambda\chi}(\phi - \mu)}_{\text{RANK AS}} \right]. \tag{23}$$

With fixed prices $\kappa = 0$ and proportional incomes $\chi = 1$, one recovers the benchmark zero-multiplier derived in RANK by Bilbiie (2011) and Woodford (2011). Positive multipliers occur with *countercyclical inequality* $\chi > 1$ through the "new Keynesian cross".[17] If the stimulus is persistent ($\mu > 0$), there is an extra kick as agents, expecting higher demand and income, also self-insure less.

The second part of Proposition 2 pertains to solving the FG puzzle. The condition is (19), i.e. determinacy under a peg, found by iterating (21) forward with $\phi = 0$ to obtain:

$$c_t = \nu_0 E_t c_{t+1} - \sigma\frac{1-\lambda}{1-\lambda\chi} i_t^* = \nu_0^{\bar{T}} E_t c_{t+\bar{T}} - \sigma\frac{1-\lambda}{1-\lambda\chi} E_t \sum_{j=0}^{\bar{T}-1} \nu_0^j i_{t+j}^*.$$

The response to a T periods ahead interest cut is, for any $T \in (t, \bar{T})$: $\frac{\partial c_t}{\partial\left(-i_{t+T}^*\right)} = \sigma\frac{1-\lambda}{1-\lambda\chi}\nu_0^T$, which is decreasing in T iff $\nu_0 < 1$. Then, since $\nu_0^{\bar{T}} E_t c_{t+\bar{T}}$ vanishes as $\bar{T} \to \infty$, solving the equation forward yields a unique solution: the earlier determinacy with a peg result. The condition $\nu_0 < 1$ rewritten as $1 - \delta > \kappa\sigma\frac{1-\lambda}{1-\lambda\chi}$ captures a simple intuition. To cure the puzzle, the left-side HANK discounting needs to dominate the right-side AS-compounding of news that is the source of trouble in RANK. This entails jointly idiosyncratic uncertainty $1 - s > 0$ and procyclical inequality $\chi < 1 - \sigma\kappa\frac{1-\lambda}{1-s} < 1$ (in oscillating THANK, this is $\chi < 1 - \frac{\sigma\kappa}{2}$). One implication is an interpretation of McKay et al (2016), where *inequality is procyclical* since profits are distributed disproportionately to low-productivity households: "as if" $\tau^D > \lambda$ here; see Hagedorn et al (2019) for a quantitative illustration.

4 Catch-22 Solutions

The foregoing analysis identifies a challenge for ensuring determinacy and avoiding puzzles in HANK with countercyclical inequality. What modelling choices can circumvent this? This section lists possible solutions: some novel to this paper, others proposed in subsequent work; they pertain to both model ingredients and to alternative policy rules.

[17] This is at work in Gali et al's (2007) early *quantitative* model, but convoluted with several other channels, and in Bilbiie and Straub (2004), Bilbiie, Meier and Mueller (2008), and Eggertsson and Krugman (2012). The last term extends the usual RANK channel: spending is inflationary; when $\phi > 1$ *real* rates go up, causing intertemporal substitution to the future.

4.1 Procyclical inequality, countercyclical risk

Proposition 2 purposefully abstracts from cyclical risk. Turning it back on can in theory resolve the Catch-22, providing amplification without the puzzle.

Proposition 3 *THANK with cyclical inequality and risk resolves the Catch-22 if and only if one channel is procyclical "enough" when the other is countercyclical; more precisely, by direct inspection of (16):*

$$\tilde{\delta} < 1 - \eta \text{ and } \frac{1-\lambda}{1-\lambda\chi} > 1 - \eta \implies \eta \in \left(\frac{(1-\chi)\,\lambda}{1-\lambda\chi}, \frac{(1-\chi)\,(1-\tilde{s})}{1-\lambda\chi} \right). \tag{24}$$

The first condition delivers discounting, while the second simultaneously yields static amplification in (16); in Appendix B I show that the latter extends to the fiscal multiplier too. The Catch-22 is thus resolved when the two channels coexist and move in opposite directions with the right relative strengths. In the procyclical-inequality case $\chi < 1$, risk needs to be countercyclical "enough" $\eta > 0$ in a way made specific by (24); in other words, the "risk channel" needs to be strong enough, as modulated by both its level and cyclicality, which depends on long-run inequality—the average consumption loss incurred when changing state.[18]

When (24) is violated it may, depending on parameter configurations, result in either (i) no puzzle, no amplification (e.g. when risk is not countercyclical enough); or (ii) amplification, but also the FG puzzle (e.g. when risk is too countercyclical). When risk and inequality are *both countercyclical* THANK delivers further amplification and aggravates the puzzle, while determinacy conditions become even more stringent, see (20). Finally, whether the Catch-22 applies also depends on whether other amplification/dampening channels operate, see below. Such other channels notwithstanding, it is important to gauge what the data suggests about the cyclicalities of inequality and risk.

A large literature measures income risk and finds that it is unambiguously *countercyclical* (suggesting $\eta > 0$): prominent examples include Storesletten et al's (2004) countercyclical-variance estimates and Guvenen et al's (2014) measure of (growth-rates) skewness cyclicality. The same is true for inequality in labor earnings: see Heathcote et al (2010) for early evidence, and Patterson (2023) for evidence controlling for MPCs, a counterpart to the model's χ, but pertaining to earnings *only*.

The picture is more nuanced for *ex-post* inequality in *disposable* income, including taxes and transfers. Since the latter are countercyclical, overall *disposable* income inequality, the true data counterpart of the model's key object, is less countercyclical and may even become *procyclical*, as illustrated in Figure 1 for the two last recessions. A recent analysis of inequality dynamics taking into account transfers is Heathcote et al (2023); their Figure 17 shows that, although earnings inequality was countercyclical as usual, *disposable* income inequality was procyclical in the COVID recession, due to unprecedented transfers to the bottom. A similar if mitigated picture emerges also for the Great Recession, especially for bottom inequality, see also Perri and Steinberg (2012).

One caveat to interpreting this evidence through the model's lens is necessary. None of the available evidence isolates and disentangles the two channels in a model-equivalent way, i.e. provide identified estimates for Γ, χ, s, and η. In that regard, the paper provides theoretical restrictions

[18] Note that for the interval to be non-empty with $\chi < 1$, we need $\lambda < 1 - \tilde{s}$. The exact condition is $\Gamma^{1/\sigma} > \frac{\lambda s}{(1-\lambda)(1-s)} = \frac{s}{1-h}$ and is always satisfied in the *oscillating THANK* limit; in the iid limit $1 - s \to \lambda$, it only requires some steady-state inequality $\Gamma > 1$. With $\chi > 1$ the condition requires procyclical risk and $1 - \tilde{s} < \lambda$.

to inform that measurement and reasons for its importance for macro transmission.[19]

4.2 Other solutions: model ingredients or policy rules

Both the Catch-22 and the FG puzzle are model properties contingent upon modelling choices but also on the Taylor rule and related to determinacy. I next review some modelling assumptions and policy rules that ensure determinacy and sidestep the Catch-22.

Model ingredients circumventing the Catch-22

One route is to abstract altogether from cyclical inequality and risk, e.g. Auclert et al (2023). This allows emphasizing other important channels that operate even in a "proportional incomes" benchmark, achieved therein by assuming sticky wages with uniform hours worked, and no profits. In section 5, I show how to blend that novel channel with cyclical inequality (this paper's focus) to gain further insights into determinacy and resolving the Catch-22.

A class of extensions of my model that resolve the Catch-22 adds deviations from full-information rational expectations, obtaining a separate source of Euler discounting that, if strong enough, can undo the compounding due to countercyclical inequality or risk. Such contributions include Pfauti and Seyrich (2023) using cognitive discounting; Elias-Gallegos (2023) using dispersed information; and Meichtry (2022) using sticky information.

Policy rules circumventing the Catch-22

Other solutions to the Catch-22 rely on determinacy with a peg and draw on results in RANK that extend to HANK. One solution is the "Wicksellian" policy rule of price level targeting which yields determinacy in RANK (Woodford (2003); Giannoni (2014)). This is especially powerful in HANK, as emphasized in Proposition 4; see the proof and discussion in Online Appendix O.C.3.

Proposition 4 ***Wicksellian rule in HANK****: In the THANK model, the Wicksellian rule $i_t = \phi_p p_t$ with $\phi_p > 0$ leads to local determinacy even when $\tilde{\delta} + \eta > 1$. Thus, the model delivers amplification without also aggravating the FG puzzle even when both inequality and risk are countercyclical.*

Intuitively, the puzzle's source is indeterminacy under a peg and a Wicksellian rule provides determinacy under a "quasi-peg": *some*, albeit small response to the price level anchors long-run expectations. A similar logic would apply to a money-supply rule, which yields price-level determinacy. This paper assumes throughout a passive-Ricardian fiscal policy. A different route to determinacy is to resort to an active, non-Ricardian fiscal rule (Leeper (1991); Woodford (1996); Cochrane (2017)). In incomplete-market economies, yet a different fiscal policy can deliver determinacy (Hagedorn (2020)); to study it in my model, we need to turn to the equilibrium with liquidity.

[19] Subsequent research quantifies the strength of each channel by estimating a DSGE version of THANK ((Bilbiie et al (2022)) or accounting Euler wedges directly (Berger et al (2023)).

5 THANK with Liquidity

In the version with liquidity in the form of government bonds used for precautionary saving, I derive the savings demand and study the intertemporal propagation and determinacy implications—with a focus on the interaction with cyclical inequality.

5.1 Savings-Liquidity Demand

Denote by B^N_{t+1} the total nominal quantity of government bonds outstanding at the end of each period. In nominal terms, $B^N_{t+1} = (1 + i_{t-1})\, B^N_t - P_t T_t$, and in real terms:

$$B_{t+1} = R_t B_t - T_t \tag{25}$$

where $R_t = \frac{1+i_{t-1}}{1+\pi_t}$ is the gross interest rate. The bond market clears $B_{t+1} = \lambda Z^H_{t+1} + (1-\lambda)\, Z^S_{t+1}$. Recall now that $Z^H_{t+1} = 0$, so that $B_{t+1} = (1-\lambda)\, Z^S_{t+1}$ and using the flow definitions:

$$B^H_{t+1} = (1-h)\, Z^S_{t+1} = \frac{1-h}{1-\lambda} B_{t+1} = \frac{1-s}{\lambda} B_{t+1},$$

and for S similarly $B^S_{t+1} = s Z^S_{t+1} = \frac{s}{1-\lambda} B_{t+1}$. The respective budget constraints imply:

$$\begin{aligned} C^H_t &= \hat{Y}^H_t + \frac{1-s}{\lambda} R_t B_t \\ C^S_t + \frac{1}{1-\lambda} B_{t+1} &= \hat{Y}^S_t + \frac{s}{1-\lambda} R_t B_t, \end{aligned} \tag{26}$$

where $\hat{Y}^j_t$ is j's disposable (net of taxes) income. Savers hold all bonds for next period; because bonds are liquid a fraction $\frac{1-s}{\lambda}$ of the payoff, including interest, accrues to next period's hand-to-mouth.

In Appendix C, I derive the steady-state demand for liquidity-bonds, or savings function:

$$B = \frac{1}{1 - R\left(1 - \frac{1-s}{\lambda}\right)} \left[\frac{1}{1 + \left(\frac{1}{\beta R} - 1\right) \frac{1-\lambda}{1-s}} - \frac{1}{1 + (1-\lambda)(\Gamma - 1)} \right], \tag{27}$$

under log utility $\sigma = 1$, normalizing $Y = 1$; (27) is an increasing convex function of R under standard restrictions. The Appendix explores this analytically in detail, but is is worth noticing that as $\beta R \to 1$ debt becomes proportional to $(1-\lambda)(\Gamma - 1)$ and thus tends to zero with no steady-state income inequality $\Gamma = 1$. The condition for a positive liquidity demand $B > 0$ is:

$$\frac{1-s}{\lambda} > 1 - \beta; \tag{28}$$

this is strictly true when $\beta R \to 1$ and implies more general restrictions on $1-s,\ \lambda,\ \beta$, see appendix C. Essentially, (28) requires *some* idiosyncratic risk and liquidity; it is violated e.g. in TANK.

5.2 Liquidity, Inequality, and Intertemporal MPCs in THANK

This version embeds a distinct amplification channel *orthogonal* to the NK Cross, the intertemporal Keynesian cross of Auclert et al (2023) (see also Hagedorn et al (2018)), and allows a novel analytical solution for their key summary statistics, the iMPCs. Loglinearizing (26) around a zero-liquidity steady state with $R = \beta^{-1}$ delivers (see Appendix C for details):

$$\begin{aligned} c_t^H &= \hat{y}_t^H + \beta^{-1}\frac{1-s}{\lambda}b_t, \\ c_t^S + \frac{1}{1-\lambda}b_{t+1} &= \hat{y}_t^S + \beta^{-1}\frac{s}{1-\lambda}b_t, \end{aligned} \tag{29}$$

where b_t is in shares of steady-state Y. Aggregating (29), we have:

$$c_t = \hat{y}_t + \beta^{-1}b_t - b_{t+1}. \tag{30}$$

The iMPCs are the partial derivatives of aggregate consumption c_t with respect to *aggregate* disposable income $\hat{y}_{t+k}$ at different horizons k, keeping fixed everything else, i.e. taxes and public debt. To find them, we solve the equilibrium dynamics of private liquid assets b_t by replacing (29) into the loglinearized self-insurance Euler equation (10); this yields the key equation relating the demand for liquid assets to individual incomes, and thus to (cyclical) income inequality:

$$E_t b_{t+2} - \Theta b_{t+1} + \beta^{-1}b_t = \frac{1-\lambda}{s}\left[sE_t\hat{y}_{t+1}^S + (1-s)\,E_t\hat{y}_{t+1}^H - \hat{y}_t^S\right], \tag{31}$$

where $\Theta \equiv \frac{1}{s} + \beta^{-1}\left[1 + \frac{1-s}{s}\left(\frac{1-s}{\lambda} - 1\right)\right]$. An important object for the model's dynamic properties—in particular, the key determinant of the iMPC's persistence—is the stable root of this (liquid-)asset accumulation equation, $x_b = \frac{1}{2}\left(\Theta - \sqrt{\Theta^2 - 4\beta^{-1}}\right)$; the general case is analyzed in Appendix C.

Here, we focus instead on distilling the role of *cyclical inequality*; the intuition is clearest in the oscillating THANK case $s = 0$, with agents saving when they expect lower income and vice versa:

$$b_{t+1} = \frac{\hat{y}_t^S - E_t\hat{y}_{t+1}^H}{2\left(1+\beta^{-1}\right)} = \frac{(2-\chi)\,\hat{y}_t - \chi E_t\hat{y}_{t+1}}{2\left(1+\beta^{-1}\right)}. \tag{32}$$

The consumption function follows by substituting this into (30), delivering Proposition 5.

$$c_t = \frac{2-\chi+\beta\chi}{2\,(1+\beta)}\hat{y}_t + \frac{2-\chi}{2\,(1+\beta)}\hat{y}_{t-1} + \frac{\beta\chi}{2\,(1+\beta)}\hat{y}_{t+1}; \tag{33}$$

Proposition 5 *The iMPCs for oscillating THANK in response to a time-T disposable income shock are:*

$$\frac{dc_T}{d\hat{y}_T} = \frac{2-\chi+\beta\chi}{2\,(1+\beta)};\ \frac{dc_{T+1}}{d\hat{y}_T} = \frac{2-\chi}{2\,(1+\beta)};\ \frac{dc_{T-1}}{d\hat{y}_T} = \frac{\beta\chi}{2\,(1+\beta)};\ \frac{dc_t}{d\hat{y}_T} = 0 \textit{ for any } t < T-1 \textit{ or } t > T+1.$$

This illustrates the key points transparently. With acyclical inequality $\chi = 1$ ($\hat{y}_t^j = \hat{y}_t$, Auclert et al's case) a current income shock induces agents to self-insure, saving in liquidity to maintain higher future consumption. While a future shock makes them consume in anticipation, depleting liquid savings. The second point concerns the additional role of the novel channel of cyclical inequality: higher income cyclicality when constrained χ makes agents consume more out of news and less out of past and current aggregate income. When self-insuring, agents take into account how the aggregate shock affects income in each state and change their asset demand and equilibrium liquidity consequently. The cyclicality of inequality thus skews the temporal path of the iMPCs and provides an additional degree of freedom for matching it. Even this simplest $s = 0$ case can then match the two key iMPCs matched by Auclert et al, the contemporaneous $dc_0/d\hat{y}_0 = 0.55$ and one-year-after $dc_1/d\hat{y}_0 = 0.15$ with $\beta = 0.95$ annually and $\chi = 1.47$.

The expressions for THANK with $s > 0$ are still analytical and convey the same intuition, but are more tedious (see Proposition 8, Appendix C). Figure 3 plots the iMPCs for THANK, expression (52) in Appendix C, along with TANK and the data from Fagereng et al. In THANK, I match the two target MPCs with $\lambda = 0.33$, $s = 0.82$ (0.96 quarterly) and $\chi = 1.4$.[20] The intertemporal path is remarkably in line with both the data and Auclert et al's quantitative HANK: the effect dies off after a few years, whereas TANK misses this intertemporal amplification altogether, the iMPCs being:

$$\frac{dc_T}{d\hat{y}_T} = \lambda\chi + (1-\lambda\chi)(1-\beta)\beta^T; \frac{dc_t}{d\hat{y}_T} = (1-\lambda\chi)(1-\beta)\beta^T \; \forall t \neq T.$$

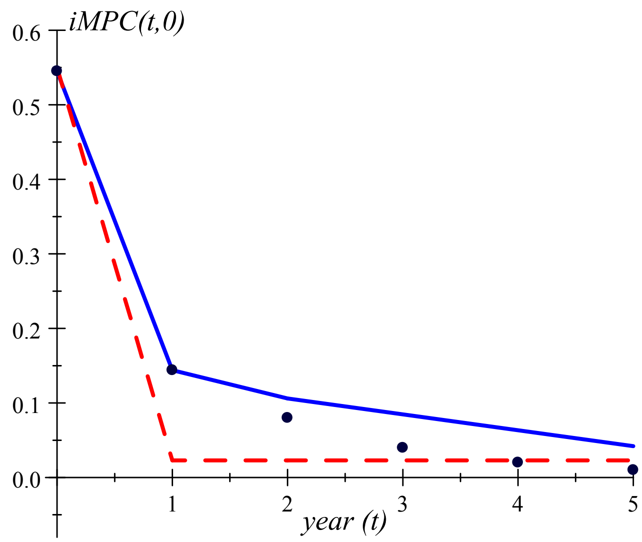

Figure 3: iMPCs in THANK (blue solid); TANK (red dash); Data (dots)

One important application of the analytical iMPCs is to clarify the connection between quantitative determinacy results and my THANK Taylor principle, underscoring again the role of cyclical inequality. In quantitative HANK, Auclert et al (2023) showed subsequently that the Taylor principle is sufficient when the sum of iMPCs (out of an income shock far into the future) is larger than 1, which makes the model "explosive" (stable forward) and thus determinate. I show this *analytically*

[20]This is coincidentally close to the calibration in Bilbiie (2020) matching *general-equilibrium* statistics with the zero-liquidity model. Figure A1 in the Appendix provides a comparison of different calibrations, and iMPCs in response to future shocks. See Cantore and Freund (2021) for a subsequent, simpler analytical iMPC calculation with portfolio adjustment costs.

by summing up the iMPCs in Proposition 5 to get:

$$\mu_{impc} = 1 + (1-\chi)\frac{1-\beta}{1+\beta} > 1 \text{ iff } \chi < 1. \tag{34}$$

This holds in the general model, see Appendix C. Thus, the requirement for Taylor principle sufficiency and for the sum of iMPCs to be larger than one is the same: procyclical inequality.

5.3 A nominal-debt rule as a Catch-22 solution

We are now ready to formulate the route to determinacy inherent in this model version (and the larger incomplete-market class of which it is an example, see Hagedorn, 2020), even in cases with countercyclical inequality and risk whereby the Taylor principle fails and the FG puzzle is normally aggravated. When the government chooses the quantity of nominal debt, the price level is determined without an interest-rate rule. If nominal taxes are set to balance the budget intertemporally for *any* price level (making policy passive-Ricardian, no fiscal theory), the central bank sets freely the nominal interest rate that clears the liquid-bond market with no need to respond to any endogenous variable. Local determinacy prevails, as shown in Proposition 6.[21]

Proposition 6 ***A nominal debt rule****: The THANK model with a well-defined demand for liquid bonds ((28) holds) leads to local determinacy even when $\delta > 1$ under the nominal-debt quantity rule:*

$$B_{t+1}^N = B^N \textit{ fixed } \rightarrow b_{t+1}^N = 0. \tag{35}$$

Intuitively, condition (28) requires that H agents receive a fraction of savings larger than the interest income, $\beta^{-1}\frac{1-s}{\lambda} > \beta^{-1} - 1 = r$; more generally, it requires that H agents receive positive net income from liquidity in steady state, see Appendix C. The proposition generalizes to $b_{t+1}^N = \phi_b p_t$ with $\phi_b < 1$ so that real debt $b_{t+1} = (\phi_b - 1)\, p_t$ falls when the price level increases. Furthermore, it can be easily shown that it holds with forward-looking Phillips curve and rules out the FG puzzle even in the "amplification" region, sidestepping the Catch-22. See Hagedorn (2020) for a general version of those arguments and Hagedorn et al (2018) for an analysis of fiscal multipliers and the earliest illustration of how a quantitative HANK with this policy rule sidesteps the Catch-22.

6 Optimal Policy in THANK

THANK is also useful for studying optimal monetary policy analytically, in the version *without liquidity*. This provides a benchmark that helps elucidate some key mechanisms operating in the rich-heterogeneity quantitative-HANK studies featuring several additional relevant channels, such as Bhandari et al (2021). I build on Woodford's (2003, Ch. 6) analysis in RANK. In Appendix D, I spell out the full Ramsey problem and derive a linear-quadratic problem equivalent to it under certain conditions, taking a second-order approximation to aggregate welfare around a flexible-price equilibrium that is *efficient*. The target equilibrium of the central bank is the socially-desirable,

[21] Given the steady-state demand for bonds (27), determinacy of the *steady-state* price level is immediate: the proof is exactly as in Hagedorn (2020). In a nutshell, monetary policy chooses steady-state i, which given π determines R and steady-state real B. The fiscal authority's choice of nominal B^N (and its growth rate) then determines P (and steady-state π).

perfect-insurance equilibrium induced by a fiscal policy generating zero profits to first order under flex prices, following the TANK analysis in Bilbiie (2008, Proposition 5). This delivers Proposition 7.

Proposition 7 *Solving the welfare maximization problem is equivalent to solving:*

$$\min_{\{c_t,\pi_t\}} \frac{1}{2} E_0 \sum_{t=0}^{\infty} \beta^t \left\{ \underbrace{\pi_t^2 + \alpha_y y_t^2}_{RANK} + \underbrace{\alpha_\gamma \gamma_t^2}_{inequality\text{-}THANK} \right\}, \tag{36}$$
s.t. (7),(12), and (13),

where the optimal weights on output and inequality stabilization are, respectively:

$$\alpha_y \equiv \left(\sigma^{-1} + \varphi\right)/\psi;\ \alpha_\gamma \equiv \lambda\left(1-\lambda\right)\alpha_y/\left(\sigma\varphi\right).$$

Several results are worth emphasizing. While the weight on output (gap) stabilization α_y is the same as in RANK, there is an additional term pertaining to (consumption or) income *inequality*.[22] This affects the central bank's stabilization tradeoff, adding a redistribution motive. However, idiosyncratic risk and its cyclicality are *irrelevant* for optimal policy, insofar as the target flexible-price equilibrium is the first-best with perfect insurance, without inequality; thus, the aggregate implications of the distributional channel for optimal policy in THANK happen to be the same as in TANK.

This is different from Challe (2020), which abstracts from inequality altogether but where an isomorphism occurs between RANK and a different analytical HANK with cyclical risk through search and matching. The common point is that my framework too features irrelevance of income risk, but in THANK relative to TANK. Both the optimal allocation and the interest rate policy instrument are radically different, and depend crucially on the cyclicality of inequality here. Furthermore, as we will see, in my framework the cyclicality risk does not matter for implementation either.

An important observation concerns the interest rate, which is both residual to the policy problem and unaffected by *risk cyclicality*. Recalling that we approximate around the efficient equilibrium $\Gamma = 1$, the IS curve (13) is *not* a constraint: as in RANK, it merely determines i_t once we found the optimal allocation (y_t, π_t). And since the IS curve approximated around $\Gamma = 1$ is also independent of cyclical risk, so will the interest rate that implements optimal policy.[23]

Consider for simplicity only shocks that drive *no wedge* between inequality and aggregate output gap, which stay proportional: (12) holds; the analysis of shocks that do drive a wedge is relevant for capturing further mechanisms in richer HANK, but beyond the scope of this paper and pursued in follow-up work. We can simplify the problem by replacing (12), obtaining the per-period loss:

$$\pi_t^2 + \alpha y_t^2, \text{ with } \alpha \equiv \alpha_y \left(1 + \frac{\lambda}{1-\lambda}\sigma^{-1}\varphi^{-1}\left(\chi - 1\right)^2\right) \tag{37}$$

[22]Other studies found additional stabilization motives using different TANK extensions, e.g. Nistico (2016), Curdia and Woodford (2016), and Benigno et al (2020) for financial-stability motives, and Bilbiie and Ragot (2021) for a liquidity-insurance motive with an imperfect-insurance target equilibrium giving rise to a *linear* term in the approximation.

[23]This is no longer the case—and risk then matters—in the model with liquidity, where the interest rate has direct distributional consequences and thus novel interactions with fiscal policy that are left to future work.

The inequality motive thus amounts, in my benchmark THANK relative to RANK, to a higher weight on output stabilization that increases with λ. Importantly, this holds regardless of whether inequality is counter- or pro-cyclical, as long as it is cyclical: the extra stabilization motive is proportional to $(\chi - 1)^2$. The simple intuition is based, as in TANK, on the key role of *profits* which are eroded by inflation volatility. With higher λ, less agents receive profits; the weight on inflation falls, and vanishes in the $\lambda \to 1$ limit, where there is no rationale for stabilizing profit income.

We can now study optimal policy in THANK starting with *discretion*, or Markov-perfect equilibrium. The ability to study this more realistic, time-consistent policy is an appealing feature of the tractable framework, since computing Markov-perfect optimal policies in quantitative models is cumbersome. This is obtained by solving (36) assuming that the central bank lacks commitment and treats expectations parametrically, without internalizing its actions' effect on them; this amounts to re-optimizing every period subject to (7) with fixed expectations at the decision time t. The problem being mathematically identical to RANK, we go directly to the solution:

$$\pi_t = -\frac{\alpha}{\kappa} y_t. \tag{38}$$

This *targeting rule* under discretion requires engineering an aggregate demand decrease for a given inflation increase. Assuming AR(1) cost-push shocks $E_t u_{t+1} = \mu u_t$, the equilibrium is:

$$\pi_t^d = \frac{\alpha}{\kappa^2 + \alpha\left(1 - \beta\mu\right)} u_t; \;\; y_t^d = -\frac{\kappa}{\kappa^2 + \alpha\left(1 - \beta\mu\right)} u_t. \tag{39}$$

Optimal policy under discretion implies that both output and inflation deviate from target: a trade-off between inflation and output stabilization. Since α is increasing in λ, it follows directly that optimal policy in THANK requires *greater* inflation and *lower* output volatility than in RANK.

One instrument rule implementing this equilibrium is found by using the aggregate IS (13):

$$i_t = \phi_d^* E_t \pi_{t+1}, \text{ with } \phi_d^* \equiv 1 + \left(\mu^{-1} - \delta\right) \frac{\kappa\left(1 - \lambda\chi\right)}{\alpha\sigma\left(1 - \lambda\right)}.$$

Unlike in RANK, the instrument rule implementing optimal policy may be *passive* $\phi_d^* < 1$ with enough compounding $\delta > \mu^{-1}$, i.e. with countercyclical enough inequality: optimal policy requires a real rate *cut* in THANK when in RANK it would require an increase. Whereas with procyclical inequality $\delta < 1$, the required instrument rule is even more active than in RANK. However, it is independent of the cyclicality of *risk* in this benchmark.

Optimal *(timeless-)commitment* policy, the time-inconsistent Ramsey equilibrium, requires committing to the different targeting rule, by similar arguments as in RANK (Woodford, 2003, Ch. 7):

$$\pi_t = -\frac{\alpha}{\kappa}\left(y_t - y_{t-1}\right). \tag{40}$$

It is straightforward to show that commitment to (40) delivers determinacy regardless of heterogeneity. The difference from RANK is still captured by the inequality motive shaping α, but optimal commitment policy still amounts to price-level targeting, like in RANK.

7 Conclusions

THANK, a tractable HANK model with two types and two assets, captures analytically several key channels of quantitative HANK models and allows elucidating their distinct and interacting roles, in particular cyclical income inequality and risk. I illustrate this by investigating these channels' roles for the dynamic properties of HANK models: determinacy, amplification, multipliers, resolving the forward guidance puzzle, and optimal monetary policy.

The key channel is *cyclical inequality*: whether the income of constrained hand-to-mouth agents comoves more or less with aggregate income. This channel is already the main focus of TANK (Bilbiie (2008)) but interacts with self-insurance and risk in THANK, as in quantitative HANK. Procyclical inequality delivers discounting in the aggregate Euler equation, which makes the Taylor principle unnecessary for determinacy and can cure the forward guidance puzzle. Conversely, countercyclical inequality generates Euler-equation compounding, making the Taylor principle insufficient for determinacy and aggravating the puzzle. This is a Catch-22, for countercyclical inequality is precisely the condition for amplification and multipliers in HANK, which is what many studies focus on, exploiting a New Keynesian cross inherent therein.

The paper proposes a decomposition of cyclical variations in income risk into one component related to cyclical inequality, and one due to cyclical skewness—variations in the probability of the bad, constrained state. The Catch-22 can be resolved if one channel is procyclical enough when the other is countercyclical—so the former delivers Euler-discounting without mitigating the amplification provided by the latter. If however both channels are countercyclical, determinacy conditions become very stringent and the puzzle is aggravated. The available evidence points to risk being countercyclical but inequality in disposable income being mildly procyclical in the last two recessions; the theory developed here can guide further measurement.

Policy rules can also circumvent the Catch-22, even when both inequality *and* risk are countercyclical. For example, a Wicksellian rule of price-level targeting resolves this tension by making THANK determinate and puzzle-free. This virtue is shared by a rule setting nominal debt proposed by Hagedorn (2020), as I show analytically in my model's version with liquidity.

Optimal monetary policy, solved for analytically in THANK, requires an inequality objective, atop stabilizing inflation and output around an efficient perfect-insurance equilibrium. Regardless of risk, optimal policy implies tolerating more inflation as a result of distributional concerns when inequality is cyclical. While timeless-optimal commitment policy still amounts to price-level targeting, even though along the adjustment path there it still entails accepting more inflation.

To date and to the best of my knowledge, THANK is the only tractable framework to capture *all* these channels found to be key in the complementary, rich-heterogeneity HANK: cyclical inequality, precautionary self-insurance saving, intertemporal MPCs, and features of idiosyncratic income uncertainty and risk (cyclical variance and skewness, and kurtosis).

As models of the economy as a whole become larger and more complex, with many sectors, frictions, and sources of heterogeneity, the quest for tractable representations seems important for entropic reasons. It is my hope that this framework is thus useful for policymakers and central banks, for communicating to the larger public, for students and colleague economists from other fields seeking to enter the fascinating realm of macro fluctuations and stabilization policy in a world where heterogeneity and inequality are of the essence.

Data availability statement: The data and code underlying this research are available on Zenodo at https://doi.org/10.5281/zenodo.10937947.

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A Related Literature

Quantitative HANK models with rich heterogeneity and feedback effects from equilibrium distributions to aggregates are increasingly used to address a wide spectrum of issues in macroeconomic policy.[24] This paper develops an *analytical* representation of the richer-heterogeneity models meant to gain insights into their mechanisms; this section reviews some of the key contributions in this literature; Online Appendix O.A.2 contains a more detailed discussion of the relation to the literature, including my own past work.

Most other analytical studies focus on the role of cyclical income risk without disentangling the role played by cyclical inequality. The clearest example is Acharya and Dogra (2020), which isolates the cyclical-risk channel by using CARA preferences to simplify heterogeneity and shows that intertemporal amplification *may* occur *purely* as a result of income volatility going up in recessions. With this different mechanism, it also studies determinacy and the forward guidance puzzle making explicit reference to the analysis in this paper's previous version.[25] Ravn and Sterk (2020) study a complementary analytical HANK with *endogenous* (through search and matching) unemployment risk and analyze determinacy and shock transmission, while Challe (2020) analyzes optimal monetary policy in that model. Werning (2015) studies the possibility of AD amplification/dampening of monetary policy in a different, general model of cyclical risk and market incompleteness, without discussing the distinction between inequality and risk and without analyzing any of the topics of this paper (determinacy with various rules, Catch-22, optimal policy).[26] Broer, Hansen, Krusell, and Oberg (2020), in another analytical HANK, show that wage rigidity can cure some of the uncomfortable implications brought about by the dynamics and distribution of profits, some of which occur in TANK in Bilbiie (2008). The companion paper Bilbiie (2020) abstracts from cyclical risk and liquidity and analyzes different issues, focusing on the important role of TANK's cyclical-inequality channel in HANK transmission in and of itself. Both that paper

[24] The effects of transfers (Oh and Reis, 2012); liquidity traps (Guerrieri and Lorenzoni, 2017); job-uncertainty-driven recessions (Ravn and Sterk, 2017; den Haan, Rendahl, and Riegler, 2018); monetary transmission (Gornemann et al, 2016; Auclert, 2018; Debortoli and Gali, 2018; Auclert and Rognlie, 2017); portfolio composition (Bayer et al, 2019; Luetticke, 2021); fiscal policy (Ferrière and Navarro, 2018, Hagedorn, Manovskii, and Mitman, 2018; Auclert et al, 2023; McKay and Reis, 2016; Cantore and Freund, 2019); the FG puzzle (McKay et al, 2016); optimal policy (Bhandari et al, 2021).

[25] Also subsequently to this paper, Auclert et al (2023) provided *numerical* determinacy results emphasizing the cyclicality of risk in quantitative HANK; Acharya and Dogra (2020) stemmed from a discussion of it meant to provide analytical insights.

[26] Holm (2021) shows that the effectiveness of monetary policy is reduced with (yet another model of) procyclical risk; see also Bernstein (2021) and Caramp and Silva (2021) for other recent analytical frameworks.

and Debortoli and Galí (2018) use the TANK version in Bilbiie (2008) to approximate some aggregate implications of *some* HANK models.

Fiscal multipliers under heterogeneity have been analyzed in several quantitative HANK models cited above and in TANK for spending (Galí et al (2007)), transfers (e.g. Bilbiie, Monacelli and Perotti (2013)) or both, in liquidity traps (Eggertsson and Krugman (2012)). Hagedorn (2020) showed that in incomplete-market economies a nominal debt quantity rule leads to price-level determinacy and rules out the FG puzzle; Hagedorn et al (2018) explored the implications for fiscal multipliers and sidestepping the Catch-22.[27] Finally, this paper is related to optimal policy studies: in RANK (i.a. Woodford (2003)) and with heterogeneity in TANKs (Bilbiie (2008); Nistico (2016); Curdia and Woodford (2016); Benigno et al (2020)). Recent different analytical HANKs providing complementary insights include Challe (2020) and Bilbiie and Ragot (2020). Important optimal-policy studies in rich-heterogeneity quantitative HANK imply deviations from price stability: Bhandari et al (2021) emphasize inequality motives, and Nuño and Thomas (2021) redistribution with nominal assets.

B Cyclical Risk in THANK

The self-insurance equation when the probability depends on aggregate demand is:

$$\left(C_t^S\right)^{-\frac{1}{\sigma}} = \beta E_t \left\{ \frac{1+i_t}{1+\pi_{t+1}} \left[s\left(Y_{t+j}\right)\left(C_{t+1}^S\right)^{-\frac{1}{\sigma}} + \left(1 - s\left(Y_{t+j}\right)\right)\left(C_{t+1}^H\right)^{-\frac{1}{\sigma}} \right] \right\}, \tag{41}$$

where $j = 0$ for current and $j = 1$ for future. We loglinearize this around a steady-state *with* inequality; this requires imperfect steady-state redistribution, i.e. the subsidy does not completely undo market power, generating zero profits. I focus on a steady state with *no* subsidy: the profit share is $D/C = 1/\varepsilon$, and $WN/C = (\varepsilon - 1)/\varepsilon$. Under the same redistribution scheme as before, the consumption shares are:

$$\begin{aligned} \frac{C^H}{C} &= \frac{WN + \frac{\tau^D}{\lambda} D}{C} = 1 - \frac{1}{\varepsilon}\left(1 - \frac{\tau^D}{\lambda}\right) \\ \frac{C^S}{C} &= \frac{WN + \frac{1-\tau^D}{1-\lambda} D}{C} = 1 + \frac{1}{\varepsilon}\frac{\lambda}{1-\lambda}\left(1 - \frac{\tau^D}{\lambda}\right) > \frac{C^H}{C} \text{ iff } \tau^D < \lambda. \end{aligned}$$

Denoting steady-state inequality $\Gamma \equiv C^S/C^H$, we loglinearize around a steady state:

$$1 = \beta\left(1 + r\right)\left[s\left(Y\right) + \left(1 - s\left(Y\right)\right)\Gamma^{\frac{1}{\sigma}}\right], \tag{42}$$

[27] Other modifications of the NK model that solve its puzzles include changing the information structure (Garcia-Schmidt and Woodford (2019), Gabaix (2019), Angeletos and Lian (2017), Farhi and Werning (2019), Woodford (2018)), pegging interest on reserves (Diba and Loisel (2017)), wealth in the utility function (Michaillat and Saez (2017), Hagedorn (2018)).

where I restrict attention to cases with positive real interest-rate r. Let r_t be the ex-ante real rate for brevity, the steady-state value of the probability $s(Y) = s$ and its elasticity relative to the cycle $-\frac{s'(Y)Y}{1-s(Y)}$.

B.1 Current aggregate demand

For the case in text (with $s(Y_t)$), the loglinearized self-insurance equation is:

$$c_t^S = -\sigma(i_t - E_t\pi_{t+1}) + \beta(1+r)sE_tc_{t+1}^S + \beta(1+r)(1-s)\Gamma^{\frac{1}{\sigma}}E_tc_{t+1}^H - \sigma\beta(1+r)\frac{s'(Y)Y}{1-s(Y)}(1-s)\left(1-\Gamma^{\frac{1}{\sigma}}\right)c_t$$

Replacing $\beta(1+r)$:

$$c_t^S = -\sigma(i_t - E_t\pi_{t+1}) + \frac{s}{s+(1-s)\Gamma^{\frac{1}{\sigma}}}E_tc_{t+1}^S + \frac{(1-s)\Gamma^{\frac{1}{\sigma}}}{s+(1-s)\Gamma^{\frac{1}{\sigma}}}E_tc_{t+1}^H + \frac{s'(Y)Y}{1-s(Y)}\frac{\sigma(1-s)\left(\Gamma^{\frac{1}{\sigma}}-1\right)}{s+(1-s)\Gamma^{\frac{1}{\sigma}}}c_t$$

Replacing c_t^S, c_t^H and using the notation for $\tilde{\delta}$, η and $\tilde{s}$ we obtain (16) in text. For the case with government spending, the aggregate Euler equation becomes:

$$c_t = \frac{\tilde{\delta}}{1-\eta}E_tc_{t+1} - \frac{\sigma}{1-\eta}\frac{1-\lambda}{1-\lambda\chi}r_t + \zeta\frac{\lambda}{1-\lambda\chi}\frac{\chi-1}{1-\eta}(g_t - E_tg_{t+1}) + \zeta\frac{\tilde{\delta}-1}{1-\eta}E_tg_{t+1} + \frac{\eta}{1-\eta}g_t$$

This can be used to compute fiscal multipliers in the case with cyclical risk, i.e. the fixed-real-rate multiplier of a one-time transitory ($E_tg_{t+1} = 0$) increase in g_t is:

$$\frac{dc_t}{dg_t} = \zeta\frac{\lambda}{1-\lambda\chi}\frac{\chi-1}{1-\eta} + \frac{\eta}{1-\eta}.$$

The first term is the NK cross, further amplified by the cyclical-risk multiplier $(1-\eta)^{-1}$; the second term is *novel*, due *exclusively* to cyclical risk. With countercyclical risk $\eta > 0$ an increase in public demand decreases risk and precautionary saving, boosting private demand with elasticity η; this generates further expansionary rounds, decreasing risk further, etc., the $(1-\eta)^{-1}$ multiplier. The amplification condition in Proposition 3 $\lambda\frac{\chi-1}{1-\lambda\chi} + \eta > 0$ is sufficient condition for a positive multiplier when $\chi < 1$, since $\zeta < 1$. When $\chi > 1$, the positive-multiplier condition is more stringent $\lambda\frac{\chi-1}{1-\lambda\chi} + \eta > \zeta\lambda\frac{\chi-1}{1-\lambda\chi} + \eta > 0$.

B.2 Future aggregate demand

For the case with $s(Y_{t+1})$, mimicking the implications of the other tractable contributions emphasizing cyclical income risk reviewed in text, the aggregate Euler-IS is:

$$c_t^S = -\sigma\left(i_t - E_t\pi_{t+1}\right) + \frac{s}{s+(1-s)\Gamma^{1/\sigma}} E_t c_{t+1}^S + \frac{(1-s)\Gamma^{1/\sigma}}{s+(1-s)\Gamma^{1/\sigma}} E_t c_{t+1}^H - \frac{s'(Y)Y}{1-s(Y)} \frac{\sigma(1-s)\left(1-\Gamma^{1/\sigma}\right)}{s+(1-s)\Gamma^{1/\sigma}} E_t c_{t+1}$$

which replacing individual consumption levels as function of aggregate becomes

$$c_t = -\sigma\frac{1-\lambda}{1-\lambda\chi}\left(i_t - E_t\pi_{t+1}\right) + \left(1 + \frac{1-\tilde{s}}{1-\lambda\chi}(\chi-1) + \frac{s'(Y)Y}{1-s(Y)}(1-\tilde{s})\frac{\sigma(1-\lambda)}{1-\lambda\chi}\left(1-\Gamma^{-1/\sigma}\right)\right) E_t c_{t+1}$$

or, using the notation in text:

$$c_t = \left(\tilde{\delta} + \eta\right) E_t c_{t+1} - \sigma\frac{1-\lambda}{1-\lambda\chi}\left(i_t - E_t\pi_{t+1}\right).$$

There is discounting if risk is procyclical enough $\eta < \frac{1-\tilde{s}}{1-\lambda\chi}(1-\chi)$. But the contemporary AD elasticity to interest rates is unaffected by the cyclicality of risk (this is thus isomorphic to Acharya and Dogra's different formalization of cyclical risk based on CARA utility).

C Liquidity in THANK

C.1 Oscillating THANK Model: Detailed Outline

This Appendix spells out the full details of the *oscillating THANK* model where agents oscillate between states. For the sake of consistency, I outline it as a special case of THANK with $s = h = 0$. An alternative representation solves the individual agent's problem (instead of a family head problem) and leads to the exact same Euler equation under the stated assumptions. The *program of the family head* is now:

$$W\left(B_t^S, B_t^H\right) = \max_{\{C_t^S, Z_{t+1}^S, C_t^H\}} \left[\frac{1}{2}U\left(C_t^S\right) + \frac{1}{2}U\left(C_t^H\right)\right] + \beta E_t W\left(B_{t+1}^S, B_{t+1}^H\right)$$

subject to the following constraints. H have a binding liquidity constraint and do not save, and S can take the asset knowing they transition next period for sure; using the same notation for beginning- and end-of-period bond holdings as in THANK, the flows from the latter into the former are:

$$B_{t+1}^S = Z_{t+1}^H = 0 \text{ and } B_{t+1}^H = Z_{t+1}^S,$$

i.e. no saving for H and all of S's end of period savings become $H's$ beginning-of period assets. The budget constraints for the respective households are thus:

$$C_t^S + Z_{t+1}^S = \hat{Y}_t^S,$$
$$C_t^H = \hat{Y}_t^H + R_{t-1} Z_t^S$$

with the same notation as before,where post-tax incomes are Y_t^j (including dividends for S); H households receive the gross real return $R_{t-1} = \frac{1+i_{t-1}}{1+\pi_t}$ on bond holdings.

The Euler equation for bonds is:

$$U'\left(C_t^S\right) = \beta E_t \left[R_t U'\left(C_{t+1}^H\right)\right]$$

S agents save in an attempt to smooth consumption relative to tomorrow's different state, where they transition with certainty. In the equilibrium with liquidity, the supply of bonds is $B_{t+1} = \frac{1}{2} Z_{t+1}^S$ and using the flow definitions, $B_{t+1}^H = Z_{t+1}^S = 2B_{t+1}$; the respective budget constraints imply:

$$\begin{aligned} C_t^H &= \hat{Y}_t^H + 2R_{t-1}B_t \\ C_t^S + 2B_{t+1} &= \hat{Y}_t^S. \end{aligned}$$

C.2 THANK: Steady-State Demand for Liquidity/Precautionary Saving

The two budget constraints (after asset market clearing) for the case with liquidity (26), evaluated at the steady state deliver, respectively, with Y^j denoting income of agent j:

$$\begin{aligned} C^H &= Y^H + \left[\left(\frac{1-s}{\lambda} - 1\right) R + 1\right] B \\ C^S &= Y^S - \frac{\lambda}{1-\lambda} \left[\left(\frac{1-s}{\lambda} - 1\right) R + 1\right] B, \end{aligned} \tag{43}$$

where we already imposed the steady-state budget $T = (R-1)B$ and assumed uniform taxation. The steady-state self-insurance Euler equation for bonds is:

$$1 = \beta R \left[s + (1-s)\left(\frac{C^S}{C^H}\right)^{\frac{1}{\sigma}}\right] \tag{44}$$

To derive the steady-state savings function (demand for liquidity), replace (43) in (44):

$$1=\beta R\left[s+(1-s)\left(\frac{Y^{S}-\frac{\lambda}{1-\lambda}\left[\left(\frac{1-s}{\lambda}-1\right)R+1\right]B}{Y^{H}+\left[\left(\frac{1-s}{\lambda}-1\right)R+1\right]B}\right)^{\frac{1}{\sigma}}\right],$$

which rewritten and under the particularly tractable case of log utility $\sigma\rightarrow 1$ becomes:

$$1=\beta R\left[1+\frac{1-s}{1-\lambda}\left(\frac{Y}{Y^{H}+\left[\left(\frac{1-s}{\lambda}-1\right)R+1\right]B}-1\right)\right]$$

Using SS inequality $\Gamma=Y^{S}/Y^{H}$ to write:

$$\frac{Y^{H}}{Y}=\frac{1}{1+(1-\lambda)(\Gamma-1)};\frac{Y^{S}}{Y}=\frac{\Gamma}{1+(1-\lambda)(\Gamma-1)}$$

we finally obtain (normalizing $Y=1$ without loss of generality) (27) in text:

$$B=\frac{1}{\left(\frac{1-s}{\lambda}-1\right)R+1}\left(\frac{1}{1+\frac{1-\lambda}{1-s}\left(\frac{1}{\beta R}-1\right)}-\frac{1}{1+(1-\lambda)(\Gamma-1)}\right).$$

Notice that SS inequality is determined in the model: using symmetric SS hours and the expression for the real wage $w=m^{-1}$ where m is the steady-state markup *with arbitrary* subsidy $m\equiv\frac{1}{1+\tau^{S}}\frac{\varepsilon}{\varepsilon-1}$, we have $Y^{H}=WN^{H}+\frac{\tau^{D}}{\lambda}D=\frac{1}{m}N+\frac{\tau^{D}}{\lambda}\frac{m-1}{m}Y=\frac{1}{m}\left(1+\frac{\tau^{D}}{\lambda}(m-1)\right)Y$ and for savers $Y^{S}=WN^{S}+\frac{1-\tau^{D}}{1-\lambda}D=\frac{1}{m}\left(1+\frac{1-\tau^{D}}{1-\lambda}(m-1)\right)Y$. Replacing both of these:

$$\Gamma=\left[1+\frac{1-\tau^{D}}{1-\lambda}(m-1)\right]/\left[1+\frac{\tau^{D}}{\lambda}(m-1)\right],$$

so there is SS inequality $\Gamma>1$ iff $m>1$ and $\tau^{D}<\lambda$.

Start by noticing that to have a self-insurance motive $C^{S}>C^{H}$ the standard condition from incomplete-market models applies: $R<\beta^{-1}$. When is SS debt positive? A positive numerator and denominator require, respectively:

$$R>\frac{1}{\beta\left[1+(1-s)(\Gamma-1)\right]}\text{ and }R<\frac{1}{1-\frac{1-s}{\lambda}},\tag{45}$$

where the latter implies that H get positive income from liquidity net of taxes ((43)). For an equilibrium interest rate to exist we thus need

$$\frac{1}{\beta\left[1+(1-s)(\Gamma-1)\right]}<\frac{1}{1-\frac{1-s}{\lambda}}\rightarrow\frac{1-s}{\lambda}>1-\beta\left[1+(1-s)(\Gamma-1)\right]$$

With large enough Γ this always satisfied as the RHS is negative $\beta^{-1}-1<(1-s)(\Gamma-1)$.

The combined condition is thus:

$$\frac{1}{\beta\left[1+(1-s)(\Gamma-1)\right]}<R<\min\left(\frac{1}{\beta},\frac{1}{1-\frac{1-s}{\lambda}}\right),$$

which in the zero-liquidity (no SS inequality) case used in the text is:

$$R=\beta^{-1}<\left(1-\frac{1-s}{\lambda}\right)^{-1}.$$

Several properties are worth noticing. As $\beta R\to 1$ debt tends to $B\to\bar{B}=\frac{1}{1+\beta^{-1}\left(\frac{1-s}{\lambda}-1\right)}\frac{(1-\lambda)(\Gamma-1)}{1+(1-\lambda)(\Gamma-1)}$ which is >0 iff $\frac{1-s}{\lambda}>1-\beta$. Moreover, $B\to 0$ when $R\to\beta^{-1}\left(1+(1-s)(\Gamma-1)\right)^{-1}$. Thus, in the "bondless", zero-liquidity limit we have (with equality in the no-SS-inequality case $\Gamma=1$):

$$R\leq\beta^{-1},$$

As standard in incomplete-market models, debt demand features an asymptote $B\to\infty$ when

$$R\to\min\left\{\beta^{-1}\frac{1}{1-\frac{1-s}{1-\lambda}},\left(1-\frac{1-s}{\lambda}\right)^{-1}\right\}.$$

First, at given numerator $B\to\infty$ when $1+R\left(\frac{1-s}{\lambda}-1\right)\to 0$ so $R\to\left(1-\frac{1-s}{\lambda}\right)^{-1}>\beta^{-1}$ under the $B>0$ requirement. Second, at given denominator, $B\to\infty$ when $1+\left(\frac{1}{\beta R}-1\right)\frac{1-\lambda}{1-s}\to 0$ so $R\to\beta^{-1}\frac{1}{1-\frac{1-s}{1-\lambda}}>\beta^{-1}$ (assuming $s>\lambda$). The former threshold is smaller than the latter when $1-\frac{1-s}{\lambda}<\beta\left(1-\frac{1-s}{1-\lambda}\right)$. Notice, however, that the threshold is always larger than β^{-1}.

An intuitive interpretation of condition (28) in text is that positive long-run liquidity, self-insurance saving requires the present discounted value of the risk of becoming constrained (infinite discounted sum of $1-s$ at β) to be larger than the *unconditional* long-run probability of being constrained.[28]

Oscillating THANK. To derive the steady-state demand for liquidity, we use post-tax income and the steady-state budget $T=(R-1)B$, assuming uniform taxation, to rewrite the budget constraints:

$$C^H=Y^H+[R+1]\,B \text{ and } C^S=Y^S-[R+1]\,B.$$

[28]Recall that the non-oscillation condition $1-s<\lambda$ stems from the law of motion $\lambda_{t+1}=(1-s)(1-\lambda_t)+h\lambda_t$ with positive root if $1-s<h\to 1-s<\lambda$.

Replacing in the steady-state self-insurance Euler equation for bonds $1 = \beta R C^S / C^H$, we have:

$$B = \frac{\beta R Y^S - Y^H}{(1+R)(1+\beta R)} \tag{46}$$

or, using SS inequality $\Gamma = Y^S / Y^H$:

$$\frac{B}{Y} = 2 \frac{\Gamma - (\beta R)^{-1}}{(1+R)\left(1 + (\beta R)^{-1}\right)(1+\Gamma)}. \tag{47}$$

C.3 Derivation of analytical iMPCs

For analytical convenience, for this section I derive results loglinearizing around a long-run steady-state with zero liquidity $B = 0$, implying $R = \beta^{-1}$ and no consumption inequality $C^H = C^S$, so (55) and the corresponding equation for S agents become (29), where we imposed asset market clearing. The equilibrium dynamics of private liquid assets b_t are found by replacing these individual budget constraints (29) into the loglinearized self-insurance Euler equation for bonds (10), with $\hat{y}_t^j \equiv y_t^j - t_t$ denoting disposable income as in text, obtaining ($\Theta \equiv \frac{1}{s} + \beta^{-1}\left[1 + \frac{1-s}{s}\left(\frac{1-s}{\lambda} - 1\right)\right]$):

$$E_t b_{t+2} - \Theta b_{t+1} + \beta^{-1} b_t = \frac{1-\lambda}{s}\left[s E_t \hat{y}_{t+1}^S + (1-s) E_t \hat{y}_{t+1}^H - \hat{y}_t^S\right]. \tag{48}$$

Finding the derivatives of b_{t+k} with respect to $\hat{y}_t$ requires a model of how individual disposable incomes are related to aggregate, such as this paper's. Furthermore, since the calculation of iMPCs keeps *fixed by definition* all the other variables (in particular taxes, their distribution, and thus public debt), the partial derivatives of individual disposable incomes with respect to aggregate disposable income are respectively $d\hat{y}_t^H = \chi d\hat{y}_t$ and $d\hat{y}_t^S = \frac{1-\lambda\chi}{1-\lambda} d\hat{y}_t$.[29] Solving the asset dynamics equation taking this into account delivers:

$$db_{t+1} = x_b db_t + \frac{1-\lambda\chi}{s} \sum_{k=0}^{\infty} (\beta x_b)^{k+1} \left(d\hat{y}_{t+k} - \delta d\hat{y}_{t+k+1}\right), \tag{49}$$

where the roots of the characteristic polynomial of (48) are $x_b = \frac{1}{2}\left(\Theta - \sqrt{\Theta^2 - 4\beta^{-1}}\right)$ and $(\beta x_b)^{-1}$, with $0 < x_b < 1$ as required by stability whenever $\beta > 1 - \frac{1-s}{\lambda}$.

Substituting (49) in (30) delivers the aggregate consumption function, the key equation for calculating the analytical iMPCs in Proposition 8:

$$dc_t = d\hat{y}_t + \beta^{-1}(1 - \beta x_b) db_t + \frac{1-\lambda\chi}{s} \sum_{k=0}^{\infty} (\beta x_b)^{k+1} \left(\delta d\hat{y}_{t+k+1} - d\hat{y}_{t+k}\right). \tag{50}$$

[29] In particular, any model would deliver a reduced-form $\hat{y}_t^H = \chi \hat{y}_t + \chi_{tax} t_t$, χ_{tax} being an equilibrium elasticity depending on the tax distribution, labor elasticity, etc. But for calculating iMPCs, we look at a partial equilibrium wherein $dt_t / \hat{y}_t = 0$.

Proposition 8 *The intertemporal MPCs (iMPCs) for the THANK model, in response to a one-time shock to disposable income at any time T and for any $t \geq 0$: (i) are given by:*

$$\frac{dc_t}{d\hat{y}_T} = \begin{cases} \frac{1-\lambda\chi}{s}\frac{\delta-\beta x_b}{1-\beta x_b^2}\left(\beta x_b\right)^{T-t}\left(1-x_b+x_b\left(1-\beta x_b\right)\left(\beta x_b^2\right)^t\right), & \textit{if} \quad t \leq T-1; \\ 1-\frac{1-\lambda\chi}{s}\beta x_b-\left(\delta-\beta x_b\right)x_b\frac{1-\lambda\chi}{s}\left(1-\beta x_b\right)\frac{1-\left(\beta x_b^2\right)^T}{1-\beta x_b^2}, & \textit{if} \quad t = T; \\ \frac{1-\lambda\chi}{s}\frac{1-\beta x_b}{1-\beta x_b^2}x_b^{t-T}\left(1-x_b\delta+x_b\left(\delta-\beta x_b\right)\left(\beta x_b^2\right)^T\right), & \textit{if} \quad t \geq T+1. \end{cases} \tag{51}$$

and (ii) are increasing with the cyclicality of inequality χ when $t < T$ and decreasing with χ when $t \geq T > 0$ (keeping fixed the time-0 contemporaneous MPC $dc_0/d\hat{y}_0$).

A useful special case occurs for $T = 0$: the iMPCs to a time-0 disposable-income shock, which are the objects plotted in Figure 3 in text, namely for any $t \geq 0$:

$$\frac{dc_t}{d\hat{y}_0} = \begin{cases} 1-\delta\frac{1-\lambda\chi}{s}+\frac{1-\lambda\chi}{s}\left(\delta-\beta x_b\right)\frac{1-x_b}{1-\beta x_b^2}, & \text{if} \quad t = 0; \\ \frac{1-\lambda\chi}{s}\left(1-\beta x_b\right)x_b^t, & \text{if} \quad t \geq 1. \end{cases} \tag{52}$$

This replicates the three key properties of both empirical and quantitative-HANK derived iMPCs: high on impact, scaled down from the second period onwards, and exponential decay thereafter. At a general level, to match these three features the model has three degrees of freedom, its deep parameters λ and s, and the cyclicality of inequality χ (at a given β).

It is useful, in order to isolate this *liquidity-amplification channel*, to follow Auclert et al's paper that discovered it and start with the benchmark of acyclical inequality $\chi = 1$. This amounts to replacing individual disposable incomes with aggregate disposable income $\hat{y}_t^j = \hat{y}_t$, obtaining the expressions in Proposition 8 with $\chi = 1$ and $\delta = 1$. The path of the iMPCs is apparent in this special case: faced with a current income shock, agents optimally self-insure, saving in liquid wealth to maintain a higher consumption in the future. While when facing a future income shock agents consume in anticipation, decreasing their stock of liquid savings. It is worth noticing that the analytical expressions thus obtained are very similar to the ones derived subsequently in the revision of Auclert et al (2023) for a different analytical heterogeneous-agent, zero-liquidity model (and hence also related, as explained therein, to their two-agent bonds-in-utility TABU model); see their Appendix D.4 and D.2, respectively.

Ceteris paribus, countercyclical inequality $\chi > 1$ leads to a higher contemporaneous MPC but to lower future MPCs (without affecting persistence as described by x_b which is independent of χ). Persistence is instead increasing with the share of hand-to-mouth and decreasing with the level of idiosyncratic risk (it can be directly verified that $\partial x_b/\partial\lambda > 0$ and $\partial x_b/\partial\left(1-s\right) < 0$).

Figure A.1 illustrates this by plotting the iMPCs for four models: TANK and three THANK cases (encompassing both liquidity and cyclical inequality) for pro- and counter-cyclical inequality, and the benchmark acyclical-inequality akin to Auclert et al's quantitative HANK, respectively. The left panel looks at a date-0 aggregate income shock and calibrates THANK with acyclical inequality to closely follow Auclert et al, i.e. $\beta = 0.8$ and $\lambda = 0.5$; this requires $s = 0.84$ to match both the contemporaneous and next-year MPCs (0.55 and 0.15). The discount rate is large, even for the yearly calibration adopted here; in the models with cyclical inequality (both TANK and THANK) I set $\beta = 0.95$ and match the two target MPCs with $\lambda = 0.33$, $s = 0.82$ and $\chi = 1.4$. This is coincidentally close to the calibration used in Bilbiie (2020) to match other (aggregate, *general-equilibrium*) objects with the same model.

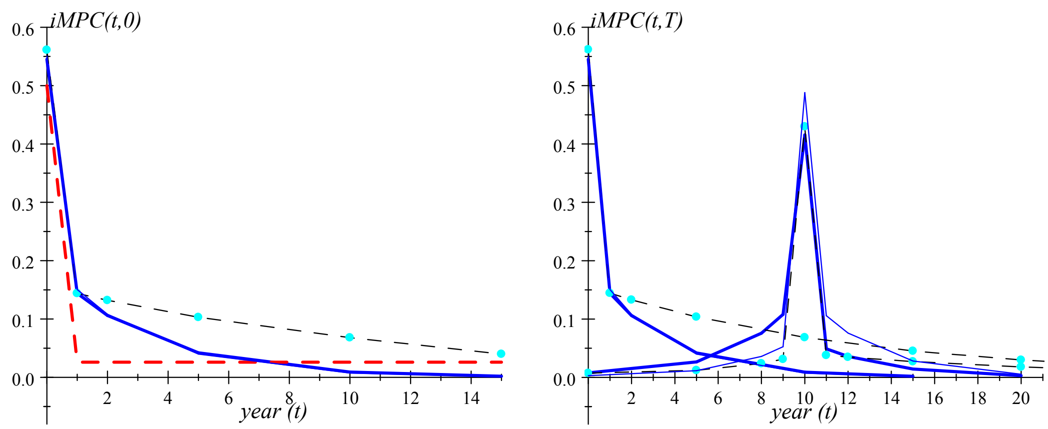

Figure A.1: iMPCs in THANK with $\chi = 1$ (thin black dot-dash); TANK (red dash); THANK with counter- and pro-cyclical inequality (thick and thin blue solid). Left: $T = 0$; right: $T = 0; 10$

The intertemporal path of the iMPCs is remarkably in line with that documented by Fagereng et al and Auclert et al in the data; in particular, the effect of the income shock dies off after a few years; whereas the model with acyclical inequality implies unrealistically high persistence while TANK implies no persistence at all. The reverse side of it is that, as clear from the right panel comparing iMPCs out of current and future income shocks for THANK with acyclical and countercyclical inequality, the latter implies larger iMPCs out of future income—an illustration of part (ii) of the Proposition; this is due intuitively to the same self-insurance forces that generate Euler-compounding. Direct differentiation of the analytical expressions in Proposition 8 reveals in fact that the iMPCs out of future income (news) are increasing in χ while the iMPCs out of past income are decreasing in χ.

An important remark is that *counter*cyclical inequality is, nevertheless, *not necessary* for the THANK model to match the iMPCs. Indeed, the model with *pro*cyclical inequality $\chi < 1$ also does it. To illustrate this, consider the model with $\chi = 0.8$. Clearly, we need to re-calibrate the model for a lower χ implies, by the logic of the cyclical-inequality channel, a lower contemporaneous MPC and a higher MPC out of past income; matching the two MPCs thus requires re-calibrating $\lambda = 0.64$ and $s = 0.74$. The resulting path (the thin solid line in the Figure) illustrates our intuition: the MPC out of past income is virtually identical, which is not surprising since we matched the one-period-ago MPC. But the whole path of the "forward" MPCs is below the countercyclical-inequality case (with the acyclical-inequality case between the two), which is a direct implication of the Euler discounting through δ discussed at length above. Notice, however, that while discounting/compounding in the Euler equation is not *per se* of the essence for matching the iMPCs (although it certainly matters quantitatively), idiosyncratic risk *is*.

C.4 Determinacy and iMPCs

Auclert et al (2019) show that determinacy occurs when the unweighted sum of iMPCs for an income shock occurring at $T \to \infty$ is larger than 1. In my model, this is calculated using Proposition 8.

$$\mu_{impc} = \lim_{T\to\infty}\left(\sum_{t=0}^{T-1}\frac{dc_t}{d\hat{y}_T} + \frac{dc_T}{d\hat{y}_T} + \sum_{t=T+1}^{\infty}\frac{dc_t}{d\hat{y}_T}\right)$$

Replacing the expressions in Proposition 8 and taking the limit for $T \to \infty$ we obtain, for the first sum:

$$\frac{1-\lambda\chi}{s}\left(\delta - \beta x_b\right)\beta x_b \frac{1-x_b}{\left(1-x_b\beta x_b\right)\left(1-\beta x_b\right)}$$

for the second term (contemporaneous iMPC):

$$1 + \left(1-\beta x_b\right)x_b\frac{1-\lambda\chi}{s}\left(\beta x_b - \delta\right)\frac{1}{1-x_b\beta x_b} - \frac{1-\lambda\chi}{s}\beta x_b$$

and for the third sum:

$$\left(1-\beta x_b\right)x_b\frac{1-\lambda\chi}{s}\frac{1}{1-x_b}\frac{1-x_b\delta}{1-x_b\beta x_b}.$$

Taking the total sum:

$$\begin{aligned}\mu_{impc} &= 1 + \frac{1-\lambda\chi}{s}x_b\left[\frac{\left(\delta-\beta x_b\right)\beta\left(1-x_b\right)}{\left(1-x_b\beta x_b\right)\left(1-\beta x_b\right)} - \beta - \frac{\left(\delta-\beta x_b\right)\left(1-\beta x_b\right)}{1-x_b\beta x_b} + \frac{1-\beta x_b}{1-x_b}\frac{1-x_b\delta}{1-x_b\beta x_b}\right] \\ &= 1 + \left(1-\delta\right)\frac{1-\lambda\chi}{s}\frac{\left(1-\beta\right)x_b}{\left(1-\beta x_b\right)\left(1-x_b\right)} \qquad (53)\end{aligned}$$

Thus, the condition for determinacy (and for the Taylor principle to be sufficient) of Auclert et al $\mu_{impc} > 1$ is equivalent to my condition $\delta < 1$.

C.5 Positive Steady-State Liquidity

In this Appendix, I derive the key loglinearized equations for the version of the model with positive steady-state liquidity—used for illustration to prove Proposition (6) in the online Appendix. The loglinearized government budget constraint is (with B_Y steady-state debt share in output and $b_t \equiv (B_t - B)/Y$):

$$b_{t+1} + t_t = R\left(b_t + B_Y i_{t-1} - B_Y \pi_t\right). \tag{54}$$

The loglinearized budget constraint of H agents (26) is:

$$\frac{C^H}{Y} c_t^H = \frac{Y^H}{Y} y_t^H - t_t + \frac{1-s}{\lambda} R b_t + \frac{1-s}{\lambda} R B_Y \left(i_{t-1} - \pi_t\right), \tag{55}$$

To derive the aggregate Euler equation for the model with liquidity and with steady-state inequality and non-zero steady-state debt, I adopt the following assumption that simplifies the algebra without losing generality: to obtain that consumption is equalized in steady-state across agents (while income is not, and there is positive debt), assume a pure redistributive transfer taken as given by agents (so it does not preclude bonds demand). Since without the transfer the share of consumption of H in Y is $C^H/Y = \left[1 + (1-\lambda)(\Gamma - 1)\right]^{-1}$, this transfer—an additive term in (43)—is trivially equal to $1 - \left[1 + \frac{1-\lambda}{1-s}\left(\frac{1}{\beta R} - 1\right)\right]^{-1}$, while for S agents we have a tax equal to the same amount times $\lambda/(1-\lambda)$. In addition, let again χ denote the equilibrium elasticity: $\frac{Y^H}{Y} y_t^H = \chi y_t$.

Under these assumptions, the budget constraint of H (55) and S become:

$$\begin{aligned}
c_t^H &= \chi y_t - t_t + \frac{1-s}{\lambda} R b_t + \frac{1-s}{\lambda} R B_Y \left(i_{t-1} - \pi_t\right) \\
c_t^S &= \frac{1}{1-\lambda} c_t - \frac{\lambda}{1-\lambda}\left(\chi y_t - t_t + \frac{1-s}{\lambda} R b_t + \frac{1-s}{\lambda} R B_Y \left(i_{t-1} - \pi_t\right)\right)
\end{aligned}$$

Replacing these in the self-insurance Euler equation we obtain (with the usual δ notation):

$$\begin{aligned}
c_t &= \delta E_t c_{t+1} - \sigma \frac{1-\lambda}{1-\lambda\chi}\left(i_t - E_t \pi_{t+1}\right) \\
&- \frac{\lambda}{1-\lambda\chi} t_t - \frac{1-\lambda-s}{1-\lambda\chi} E_t t_{t+1} + \frac{1-\lambda-s}{1-\lambda\chi}\frac{1-s}{\lambda} R b_{t+1} + \frac{\lambda}{1-\lambda\chi}\frac{1-s}{\lambda} R b_t \\
&+ \frac{1-\lambda-s}{1-\lambda\chi}\frac{1-s}{\lambda} R B_Y \left(i_t - E_t \pi_{t+1}\right) + \frac{\lambda}{1-\lambda\chi}\frac{1-s}{\lambda} R B_Y \left(i_{t-1} - \pi_t\right)
\end{aligned} \tag{56}$$

This is the key equation used for the determinacy proof of Proposition (6) in the online Appendix.

D Optimal Policy in THANK: Ramsey and a Second-Order Approximation

First, we write explicitly the Ramsey problem, and then we derive the second-order approximation around an efficient equilibrium that allows transforming it into a linear-quadratic problem.

The **Ramsey problem in THANK,** of maximizing a utilitarian welfare objective, is:

$$\max_{\{C_t^H,C_t^S,N_t^H,N_t^S,\pi_t\}} E_0 \sum_{t=0}^{\infty} \beta^t \{\lambda U\left(C_t^H, N_t^H\right) + (1-\lambda)\, U\left(C_t^S, N_t^S\right) \tag{57}$$
$$+\varsigma_{1,t}\left(\frac{U_N\left(N_t^S\right)}{U_C\left(C_t^S\right)} - \frac{U_N\left(N_t^H\right)}{U_C\left(C_t^H\right)}\right)$$
$$+\varsigma_{2,t}\left(C_t^H + \frac{U_N(N_t^H)}{U_C(C_t^H)} N_t^H - \frac{\tau^D}{\lambda}(1-\frac{\psi}{2}\pi_t^2 + \frac{U_N(N_t^H)}{U_C(C_t^H)})(\lambda N_t^H + (1-\lambda)N_t^S)\right)$$
$$+\varsigma_{3,t}\left(\lambda C_t^H + (1-\lambda)C_t^S - (1-\frac{\psi}{2}\pi_t^2)(\lambda N_t^H + (1-\lambda)N_t^S)\right)$$
$$+\varsigma_{4,t}\{\pi_t(1+\pi_t) - \beta E_t[\frac{U_C(C_{t+1}^S)}{U_C(C_t^S)} \frac{\lambda N_{t+1}^H + (1-\lambda)N_{t+1}^S}{\lambda N_t^H + (1-\lambda)N_t^S} \pi_{t+1}(1+\pi_{t+1})] + \frac{\varepsilon-1}{\psi}[\frac{\varepsilon}{\varepsilon-1}\frac{U_N(N_t^H)}{U_C(C_t^H)} + 1 + \tau^S]\}$$

where $\varsigma_{j,t}$ the co-state Lagrange multipliers associated to them (with arbitrary initial values).

In the above Ramsey constraints, we already substituted $C_t = \left(1 - \frac{\psi}{2}\pi_t^2\right) Y_t = \left(1 - \frac{\psi}{2}\pi_t^2\right) N_t ...$, $W_t = -U_N\left(N_t^S\right)/U_C\left(C_t^S\right) = -U_N\left(N_t^H\right)/U_C\left(C_t^H\right)$, and eliminated $D_t = \left(1 - \frac{\psi}{2}\Pi_t^2 - W_t\right)\left(\lambda N_t^H + (1-\lambda)\, N_t^S\right)$

Importantly, notice that the self-insurance equation is not a constraint—just as in RANK the Euler-IS curve is not a constraint. In other words, the equation

$$U_C(C_t^S) = \beta E_t\left[\frac{1+i_t}{1+\pi_{t+1}}\left(s(C_{t+1})U_C(C_{t+1}^S) + (1-s(C_{t+1}))U_C(C_{t+1}^H)\right)\right]$$

determines i_t residually once we found the allocation.[30]

Note that it is trivial to show that the *first-best* equilibrium amounts to perfect insurance. And solving the above Ramsey problem and finding the optimal steady-state inflation can be easily shown to deliver long-run price stability ($\pi = 0$) as the optimal long-run target.

A Second-Order Approximation to Welfare. We approximate the economy around an efficient equilibrium, defined as an equilibrium with both flexible prices and perfect insurance; this is the case in our baseline economy under the assumed steady-state fiscal policy, because the optimal subsidy inducing zero profits in steady state implies that consumption shares are equalized across agents. In particular, since the fiscal authority subsidize sales at the constant rate τ^S and redistribute the proceedings in a lump-sum fashion T^S such that in steady-state there is marginal cost pricing, and profits are zero. The profit function becomes $D_t(k) = \left(1+\tau^S\right) P_t(k)Y_t(k) - W_tN_t(k) - \frac{\psi}{2}\left(\frac{P_t(k)}{P_{t-1}^{**}} - 1\right)^2 P_tY_t + T_t^S$ where by balanced budget $T_t^S = \tau^S P_t(k)Y_t(k)$. Efficiency requires $\tau^S = (\varepsilon-1)^{-1}$, such that under flexible prices $P_t^*(k) = W_t^*$ and hence profits are $D_t^* = 0$ (evidently, with sticky prices profits are not zero as the

[30]The interest rate is no longer orthogonal in models where it affects the mapping between the income and consumption distribution and aggregate income, for instance because it determines the MPC as in the Acharya and Dogra's PRANK model. See Acharya, Challe, and Dogra (2023) for a recent exploration of the optimal policy implications of this.

mark-up is not constant). Under this assumption we have that in steady-state:

$$\frac{U_N\left(N^H\right)}{U_C\left(C^H\right)}=\frac{U_N\left(N^S\right)}{U_C\left(C^S\right)}=\frac{W}{P}=1=\frac{Y}{N},$$

where $N^j=N=Y$ and $C^j=C=Y$.

Suppose further that the social planner maximizes a convex combination of the utilities of the two types, weighted by the mass of agents of each type: $U_t(.)\equiv\lambda U^H\left(C_t^H,N_t^H\right)+[1-\lambda]\,U^S\left(C_t^S,N_t^S\right)$. The second-order approximation to type $j's$ utility around the **efficient flex-price equilibrium** delivers:

$$\begin{aligned}\hat{U}_{j,t} &\equiv U_j\left(C_{j,t},N_{j,t}\right)-U_j\left(C_{j,t}^*,N_{j,t}^*\right)=\\ &= U_C C^j\left[c_t^j+\frac{1-\sigma^{-1}}{2}\left(c_t^j\right)^2\right]-U_N N^j\left[n_t^j+\frac{1+\varphi}{2}\left(n_t^j\right)^2\right]+t.i.p+O\left(\parallel\zeta\parallel^3\right),\end{aligned}\tag{58}$$

where we used that flex-price values are equal to steady-state values (because of our assumption of no shocks to the natural rate) $c_t^{j*}\left(\equiv\log\frac{C_t^{j*}}{C}\right)=c_t^*=0$ and $n_t^{j*}\left(\equiv\log\frac{N_t^{j*}}{N}\right)=n_t^*=0$.

Approximating the goods market clearing condition to second order delivers:

$$\begin{aligned}\lambda C_{H,t}+(1-\lambda)\,C_{S,t} &\simeq \lambda c_{H,t}+(1-\lambda)\,c_{S,t}+\frac{1}{2}\left(\lambda c_{H,t}^2+(1-\lambda)\,c_{S,t}^2\right)\\ &= \lambda N_{H,t}+(1-\lambda)\,N_{S,t}\simeq\lambda n_{H,t}+(1-\lambda)\,n_{S,t}+\frac{1}{2}\left(\lambda n_{H,t}^2+(1-\lambda)\,n_{S,t}^2\right)\end{aligned}$$

The linearly-aggregated first-order term is thus found from this second-order approximation of the economy resource constraint as:

$$\lambda c_{H,t}+(1-\lambda)\,c_{S,t}-\lambda n_{H,t}-(1-\lambda)\,n_{S,t}+\frac{1}{2}\left(\lambda c_{H,t}^2+(1-\lambda)\,c_{S,t}^2-\left(\lambda n_{H,t}^2+(1-\lambda)\,n_{S,t}^2\right)\right)=0\tag{59}$$

The economy resource constraint is

$$C_t=\left(1-\frac{\psi}{2}\pi_t^2\right)Y_t=\left(1-\frac{\psi}{2}\pi_t^2\right)N_t$$

which approximated to second order is:

$$c_t=n_t-\frac{\psi\pi}{1-\frac{\psi}{2}\pi^2}\pi_t-\frac{1}{2}\frac{\psi}{1-\frac{\psi}{2}\pi^2}\pi_t^2$$

It is straightforward to show that the optimal long-run inflation target in this economy is, just like in

RANK, $\pi = 0$. Replacing, we obtain the second-order approximation of the resource constraint:

$$c_t = n_t - \frac{\psi}{2}\pi_t^2, \tag{60}$$

where the second term captures the welfare cost of inflation around the optimal long-run equilibrium.

Note that since $U_C C^j$ and $U_N N^j$ are equal across agents we can aggregate the approximations of individual utilities above (58), using (59) and (60) to eliminate linear terms, into:

$$\hat{U}_t = -U_C C\left\{\frac{\sigma^{-1}}{2}\left[\lambda\left(c_t^H\right)^2 + (1-\lambda)\left(c_t^S\right)^2\right] + \frac{\varphi}{2}\left[\lambda\left(n_t^H\right)^2 + (1-\lambda)\left(n_t^S\right)^2\right] + \frac{\psi}{2}\pi_t^2\right\} + t.i.p + O\left(\| \zeta \|^3\right).$$

Quadratic terms can be expressed as a function of aggregate consumption (output). Notice that in evaluating these quadratic terms we can use first-order approximations of the optimality conditions (higher order terms imply terms of order $O\left(\| \zeta \|^3\right)$). Recall that up to first order, we have that $c_t^H = \chi y_t$ and $c_t^S = \frac{1-\lambda\chi}{1-\lambda} y_t$ and (after straightforward manipulation for hours worked):

$$n_t^H = \left(1 + \varphi^{-1}\sigma^{-1}(1-\chi)\right) y_t \text{ and } n_t^S = \left(1 + \varphi^{-1}\sigma^{-1}\frac{\lambda}{1-\lambda}(\chi - 1)\right) y_t$$

To second order we thus have

$$\begin{aligned}
\left(c_t^H\right)^2 &= \chi^2 y_t^2 + O\left(\| \zeta \|^3\right); \ \left(n_t^H\right)^2 = \left[1 + \varphi^{-1}\sigma^{-1}(1-\chi)\right]^2 y_t^2 + O\left(\| \zeta \|^3\right) \\
\left(c_t^S\right)^2 &= \left(\frac{1-\lambda\chi}{1-\lambda}\right)^2 y_t^2 + O\left(\| \zeta \|^3\right); \ \left(n_t^S\right)^2 = \left[1 + \varphi^{-1}\sigma^{-1}\frac{\lambda}{1-\lambda}(\chi-1)\right]^2 y_t^2 + O\left(\| \zeta \|^3\right)
\end{aligned}$$

Replacing in the per-period welfare, considering only the "real" part (abstracting from inflation) for notational convenience and ignoring terms independent of policy and of order larger than 2:

$$\begin{aligned}
&\frac{\sigma^{-1}}{2}\left[\lambda\left(\chi c_t\right)^2 + (1-\lambda)\left(\frac{1-\lambda\chi}{1-\lambda}c_t\right)^2\right] \\
&+\frac{\varphi}{2}\left[\lambda\left(\left[1 + (\varphi\sigma)^{-1}(1-\chi)\right]c_t\right)^2 + (1-\lambda)\left(\left[1 - \frac{\lambda}{1-\lambda}(\varphi\sigma)^{-1}(1-\chi)\right]c_t\right)^2\right]
\end{aligned}$$

Collecting terms, multiplying by 2 for simplicity, and rearranging, using $c_t = y_t$:

$$
\begin{aligned}
& \sigma^{-1}\left(1+\frac{\lambda}{1-\lambda}(\chi-1)^2\right)y_t^2+\varphi\left(1+\frac{\lambda}{1-\lambda}\left[(\varphi\sigma)^{-1}(\chi-1)\right]^2\right)y_t^2 \\
= & \left(\sigma^{-1}+\varphi+\sigma^{-1}\frac{\lambda}{1-\lambda}(\chi-1)^2\left(1+(\varphi\sigma)^{-1}\right)\right)y_t^2 \\
= & \left(\sigma^{-1}+\varphi\right)y_t^2+\sigma^{-1}\frac{\lambda}{1-\lambda}\left(1+(\varphi\sigma)^{-1}\right)\left[(\chi-1)\,y_t\right]^2 \\
= & \left(\sigma^{-1}+\varphi\right)\left[y_t^2+\sigma^{-1}\varphi^{-1}\lambda(1-\lambda)\,\gamma_t^2\right]
\end{aligned}
$$

where the last equality used the expression for γ_t (12). Adding back the inflation term, we obtain the loss function in (36) in Proposition 7 in text.

Online Appendix to

Monetary Policy and Heterogeneity: An Analytical Framework

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Abstract: This Online Appendix contains derivations and other details accompanying "Monetary Policy and Heterogeneity: An Analytical Framework".

O.A Data, Model and Literature Details

This Appendix presents in detail the data and model and reviews the connection to the literature.

O.A.1 Data: Details

All data is from realtimeinequality.org, see Blanchet, Saez, and Zucman (2023) for details and a complementary detailed discussion of the COVID recession. Total Factor Income includes *all* labor and capital income before taxes; Disposable Income is income net of all taxes and including all cash government transfers and food stamps. I use the "real income growth" for all series, and look at households. The samples are December 2007 to July 2009, and January 2020 to August 2020. respectively; they are chosen to just include the respective recessions according to the NBER definition.

I construct the income growth series for the top 50% by aggregating the top 10% and the middle 40% provided; for each subsample, I use the relative income shares in the first period therein when constructing the aggregate growth rate. The Figure reports the difference between the growth rate of the top 50% thus constructed, and the directly available bottom 50%. The Figure is similar (if less stark) when computing inequality between the middle 40 and the bottom 50, as shown in Figure OA1.

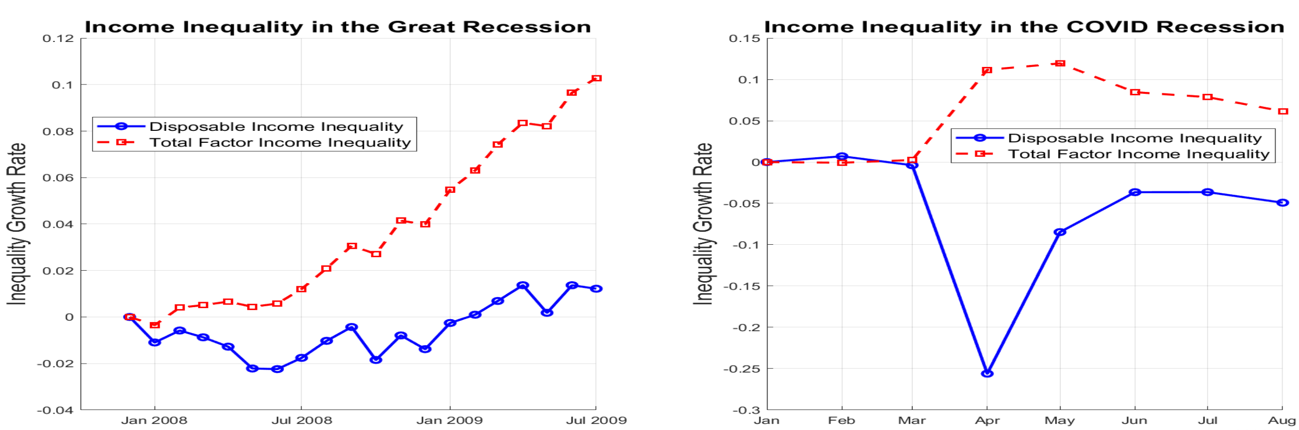

Fig. OA1: Income inequality (middle 40/bottom 50%) in the last two recessions; realtimeinequality.org data (Blanchet et al, 2023)

O.A.2 Relation to Literature: Details

Relation to other analytical HANK Others studies also provide different analytical frameworks, both because they isolate *different* HANK mechanisms and focus on different questions. The clearest separation in terms of channels is illustrated by the subsequent paper by Acharya and Dogra (2020) reviewed in text, that is explicitly set to isolate cyclical risk using CARA preferences. That paper shows that indeed intertemporal amplification *may* occur *purely* as a result of uninsurable income volatility going up in recessions, even when inequality is acyclical. (the paper also studies determinacy and puzzles referring to this paper's results from the previous version.) In a previous contribution, Werning (2015) similarly emphasizes the possibility of AD amplification/dampening of monetary policy relative to RANK in a more general model of income risk and market incompleteness where inequality and risk coexist. My paper's subject is very different, a full analysis of NK topics. So is the mechanism, although some of its equilibrium implications pertaining to intertemporal amplification or dampening have a similar flavor. But the key here is *cyclical inequality:* the *distribution* of income (between labor and "capital" understood as monopoly profits) and how it depends on aggregate income, as summarized through χ, the chief feature of my earlier TANK model Bilbiie (2008). Whereas the discussion in Werning emphasizes the *cyclicality of income risk*: as uninsurable income risk goes up in a recession, agents increase their precautionary savings and decrease their consumption, amplifying the initial recession which further increases idiosyncratic risk, and so on—a mechanism previously emphasized through endogenous unemployment risk by Ravn and Sterk (2020) and Challe et al (2020). My model's mechanism is instead an *intertemporal* extension of the cornerstone amplification (dampening) mechanism in TANK, when any agent can become constrained in any future period and self-insures (imperfectly) using liquid assets against the (acyclical) risk of doing so. This puts the *cyclicality of income of constrained*, and thus of inequality, at the core of transmission; whereas Werning emphasizes the cyclicality of income *risk*, although the two *are* convoluted in the different, more general framework therein.

To incorporate this distinction, I embed a separate cyclical-risk channel in THANK, assuming that the probability of becoming constrained is a function of aggregate output. With this different formalization, the two different channels of cyclical inequality and risk jointly determine AD amplification. Not only *are* the two channels naturally separate: my analysis implies that they *better be distinct*, for in order to resolve the Catch-22 they *need* to go in opposite directions. Which channel prevails empirically is a very interesting and hitherto unexplored topic that I pursue currently.

Additionally, my analysis is conducted in a loglinearized NK model that nests not only the three-equation textbook RANK but also: TANK, a HANK with cyclical inequality and acyclical risk, and a HANK with cyclical risk and acyclical inequality. Since it is so simple and transparent and close to standard NK craft, it may be of independent interest to some researchers.

My results imply an analytical reinterpretation of McKay et al's (2016, 2017) incomplete-markets based resolution of the FG puzzle. My framework underscores the *procyclicality of inequality* as sufficient for delivering Euler-equation discounting in the presence of (albeit *acyclical*) idiosyncratic risk. Procyclicality of inequality occurs in my model through labor market features and fiscal redistribution making the income of constrained agents vary less than one-to-one with the cycle $\chi < 1$. If inequality is instead *countercyclical*, the Euler equation is *compounded* in my model, implying an aggravation of the FG puzzle. Furthermore, my paper addresses a wide range of NK topics as mentioned above.

Broer, Hansen, Krusell, and Oberg (2020) study a simplified HANK whose equilibrium has a two-agent representation, underscoring the implausibility of some of the model's implications for monetary transmission through income effects of profit variations on labor supply—and showing that a sticky-wage version features a more realistic transmission mechanism; Walsh (2017) provides another analytical model with heterogeneity emphasizing the role of sticky wages (see Colciago (2011), Ascari, Colciago, and Rossi (2017), and Furlanetto (2011) for earlier sticky-wage TANK).

Auclert, Rognlie, and Straub (2018) also use a "Keynesian cross" version to capture a distinct, complementary HANK channel. In particular, they abstract from the cyclical-inequality channel emphasized here to focus on the role of *liquidity* in the form of public debt; they unveil key summary statistics pertaining to the marginal propensities to consume out of past and future income (labelled iMPCs) and how they shape the responses of the economy to past and future income shocks. Their quantitative HANK model with liquid and illiquid assets can in fact be viewed as the closest generalization of my THANK model; or alternatively, it is among the wide spectrum of quantitative HANK models the one to which my THANK model is the closest reduced representation. I use THANK to calculate analytically Auclert et al's iMPCs and provide insights into the important propagation mechanism they emphasize. Indeed, self-insurance to idiosyncratic risk is necessary and sufficient in the presence of liquidity to generate the tent-shaped path of iMPCs in THANK; whereas cyclical inequality is not of the essence to generate persistent iMPCs, but is important to fit the magnitudes under realistic calibrations.

Ravn and Sterk (2020) also study an analytical HANK but with search and matching (SaM), that is different from and complementary to my model and focusing on a different (sub)set of the issues studied here; Challe (2020) studies optimal monetary policy therein. Their models include *endogenous* unemployment risk through labor SaM, risk against which workers self-insure. The simplifying assumptions used to maintain tractability, in particular pertaining to the asset market, are orthogonal to mine.[31] Their

[31] In my model savers hold, price, and receive the payoff (profits) of shares. In Ravn and Sterk, hand-to-mouth workers get the return on shares but do not price them. Their mechanism creates an "unemployment-trap", a breakup of the Taylor principle complementary to the one here, and fixes the puzzling NK effects of supply shocks in a LT, which I abstract from.

framework delivers an interesting feedback loop from precautionary saving to aggregate demand (see also Challe et al (2017)). My benchmark model does much the opposite: in the zero-liquidity case, it gains tractability assuming *exogenous* transitions and a different asset market structure, but emphasizes the NK-cross feedback loop through the *endogenous* constrained income that is absent in Ravn and Sterk and Challe. While my extension to cyclical risk can be viewed as a reduced-form formalization of their channel. This paper addresses additional topics: forward guidance and the FG puzzle, (restoring determinacy under a peg), the Catch-22 and a way out of it, the virtues of a Wicksellian rule of price-level targeting, the version with liquidity, and optimal monetary policy.

Relation to Bilbiie (2020) and (2008) The THANK model proposed here is an extension of the TANK model in Bilbiie (2008), which analyzed monetary policy introducing the distinction between the two types based on *asset markets participation*:[32] H have no assets, while S own all the assets, i.e. price bonds and shares in firms through their Euler equation. That paper analyzed AD amplification of monetary policy and emphasized the key role of *profits* and their distribution, as well as of fiscal redistribution, for this in an analytical 3-equation TANK model isomorphic to RANK. In recent work, Bilbiie (2020) and Debortoli and Galí (2018) both used this TANK model to argue that it can approximate reasonably well some aggregate implications of *some* HANK models: several models from the HANK literature cited above, for the former; and the authors' own, for the latter. This suggests that the cyclical-inequality channel plays an important role in HANK transmission in and of itself.

The first extension here pertains to introducing self-insurance to idiosyncratic uncertainty: the risk of becoming constrained in the future despite not being constrained today, a key HANK mechanism that is absent in TANK; this gives the model another margin to fit the aggregate findings of quantitative HANK, as shown in Bilbiie (2020).[33] That paper introduces the New Keynesian Cross as a graphical and analytical apparatus for the AD side of HANK models, expressing its key objects—MPC and multipliers—as functions of heterogeneity parameters. It studies the implications for monetary and fiscal multipliers, the link between MPC and multipliers with the "direct-indirect" decomposition of Kaplan et al, and the ability of this simple model to replicate some aggregate equilibrium implications of several quantitative, micro-calibrated HANK models. Finally, Bilbiie and Ragot (2021) builds a different analytical HANK with three assets—one ("money") liquid and traded in equilibrium, two (bonds and stock) illiquid—and studies Ramsey-optimal monetary policy as liquidity provision.

[32] Thus abstracting from *physical investment*, the element of distinction in previous two-agent studies: Mankiw (2000) had used a growth model with this distinction, due to pioneerig work by Campbell and Mankiw (1989), to analyze long-run fiscal policy issues. Galí, Lopez-Salido and Valles (2007) embedded this same distinction in a NK model and studied numerically the business-cycle effects of government spending, with a focus on obtaining a positive multiplier on private consumption. They also analyzed numerically determinacy properties of interest rate rules, that Bilbiie (2008) derived analytically.

[33] That paper also discusses the differences with earlier work using type-switching to analyze monetary policy, e.g. Nistico (2016) and Curdia and Woodford (2016). I spell out the differentiating assumptions below.

This paper's novel elements include: adding cyclical risk from several sources, related or unrelated to inequality, and pertaining to either variance or skewness; liquidity and a calculation of the iMPCs; an aggregate supply side and closed-form conditions for determinacy with Taylor rules (the HANK Taylor principle), for determinacy under price-level targeting, and for ruling out the forward-guidance puzzle; a formal statement of the "Catch-22" and of the conditions on the cyclicalities of risk and inequality to rule it out; an analysis of optimal monetary policy.

O.A.3 Aggregate Supply: New Keynesian Phillips Curve

All households consume an aggregate basket of individual goods $k \in [0,1]$, with constant elasticity of substitution $\varepsilon > 1$: $C_t = \left(\int_0^1 C_t(k)^{(\varepsilon-1)/\varepsilon} dk\right)^{\varepsilon/(\varepsilon-1)}$ yielding the standard demand $C_t(k) = (P_t(k)/P_t)^{-\varepsilon} C_t$ and aggregate price index $P_t^{1-\varepsilon} = \int_0^1 P_t(k)^{1-\varepsilon} dk$. Good are produced by monopolistic firms using labor: $Y_t(k) = N_t(k)$, with real marginal cost W_t.

The profit function is: $D_t(k) = \left(1+\tau^S\right)[P_t(k)/P_t] Y_t(k) - W_t N_t(k) - T_t^F$.

The individual goods producers solve:

$$\max_{P_t(k)} E_0 \sum_{t=0}^{\infty} Q_{0,t}^S \left[\left(1+\tau^S\right) P_t(k) Y_t(k) - W_t N_t(k) - \frac{\psi}{2}\left(\frac{P_t(k)}{P_{t-1}^{**}} - 1\right)^2 P_t Y_t\right],$$

where I consider two possibilities for the reference price level P_{t-1}^{**}, with respect to which it is costly for firms to deviate. In the first scenario, this is the aggregate price index P_{t-1} which small atomistic firms take as given—this delivers the static Phillips curve. In the second, P_{t-1}^{**} is firm k's own individual price as in standard formulations. $Q_{0,t}^S \equiv \beta^t \left(P_0 C_0^S / P_t C_t^S\right)^{\sigma^{-1}}$ is the marginal rate of intertemporal substitution of participants between times 0 and t, and τ^S the sales subsidy. Firms face demand for their products from two sources: consumers and firms themselves (in order to pay for the adjustment cost); the demand function for the output of firms z is $Y_t(k) = (P_t(k)/P_t)^{-\varepsilon} Y_t$. Substituting this into the profit function, the first-order condition is, after simplifying, for each case:

Static PC case $P_{t-1}^{**} = P_{t-1}$

$$0 = Q_{0,t}\left(\frac{P_t(k)}{P_t}\right)^{-\varepsilon} Y_t \left[\left(1+\tau^S\right)(1-\varepsilon) + \varepsilon \frac{W_t}{P_t}\left(\frac{P_t(k)}{P_t}\right)^{-1}\right] - Q_{0,t}\psi P_t Y_t \left(\frac{P_t(k)}{P_{t-1}} - 1\right)\frac{1}{P_{t-1}}$$

In a symmetric equilibrium all producers make identical choices (including $P_t(k) = P_t$); defining net inflation $\pi_t \equiv (P_t/P_{t-1}) - 1$, this becomes:

$$\pi_t (1+\pi_t) = \frac{\varepsilon - 1}{\psi}\left[\frac{\varepsilon}{\varepsilon - 1} w_t - \left(1+\tau^S\right)\right],$$

loglinearization of which delivers the static PC in text (8).[34]

Dynamic PC case $P^{**}_{t-1} = P_{t-1}$; the first-order condition is

$$0 = Q_{0,t}\left(\frac{P_t(k)}{P_t}\right)^{-\varepsilon} Y_t\left[\left(1+\tau^S\right)(1-\varepsilon)+\varepsilon\frac{W_t}{P_t}\left(\frac{P_t(k)}{P_t}\right)^{-1}\right]$$
$$-Q_{0,t}\psi P_t Y_t\left(\frac{P_t(k)}{P_{t-1}(k)}-1\right)\frac{1}{P_{t-1}(k)}+E_t\left\{Q_{0,t+1}\left[\psi P_{t+1}Y_{t+1}\left(\frac{P_{t+1}(k)}{P_t(k)}-1\right)\frac{P_{t+1}(k)}{P_t(k)^2}\right]\right\}$$

In a symmetric equilibrium, using again the definition of net inflation π_t, and noticing that $Q_{0,t+1} = Q_{0,t}\beta\left(C^S_t/C^S_{t+1}\right)^{\sigma^{-1}}(1+\pi_{t+1})^{-1}$, this becomes:

$$\pi_t(1+\pi_t) = \beta E_t[(\frac{C^S_t}{C^S_{t+1}})^{1/\sigma}\frac{Y_t}{Y_{t+1}}\pi_{t+1}(1+\pi_{t+1})]+\frac{\varepsilon}{\psi}\left(w_t-\frac{1}{m}\right),$$

with the post-subsidy markup $m \equiv \frac{1}{1+\tau^S}\frac{\varepsilon}{\varepsilon-1}$. Loglinearizing this delivers the NKPC in text (7); notice that this nests the static PC when the discount factor of firms $\beta = 0$.

I assume as a benchmark that the government implements the standard NK optimal subsidy inducing marginal cost pricing $m = 1$, i.e. $\tau^S = (\varepsilon-1)^{-1}$. Financing its total cost by taxing firms $T^F_t = \tau^S Y_t$ gives total profits $D_t = Y_t - W_t N_t$. This policy is *redistributive*: since steady-state profits are zero $D = 0$, it taxes the firms' shareholders and results in the "full-insurance" steady-state used here as a benchmark $C^H = C^S = C$. Loglinearizing around this and denoting $d_t \equiv \ln(D_t/Y)$ we have $d_t = -w_t$: profits vary inversely with the real wage an extreme form of the general property of NK models.

O.B Income Processes, Inequality, and Cyclical Idiosyncratic Risk

O.B.1 Inequality, Gini Coefficient, and Generalized Entropy

This section discusses the relationship between our measure of inequality Γ_t and the more standard measures: first, Gini coefficient, and then generalized entropy.

The **income Gini** with two levels is given by

$$\Phi_t = \frac{(1-\lambda)Y^S_t}{Y_t}-(1-\lambda) = (1-\lambda)\left(\frac{Y^S_t}{Y_t}-1\right)$$

[34]In a Calvo setup, this amounts to assuming that each period a fraction of firms f keep their price fixed, while the rest can re-optimize freely *but* ignoring that this price affects future demand. This reduces to $\beta_f = 0$ *only* in the firms' problem (not recognizing that today's reset price prevails with some probability in future periods).

and is between 0 and λ (when S get all income). Rewrite it using our measure as

$$\Phi_t = (1-\lambda)\left(\frac{\Gamma_t}{\lambda + (1-\lambda)\Gamma_t} - 1\right) = \lambda \frac{(1-\lambda)(\Gamma_t - 1)}{1 + (1-\lambda)(\Gamma_t - 1)}$$

and conversely $\Gamma_t = 1 + \frac{\Phi_t}{(\lambda - \Phi_t)(1-\lambda)}$. Using the log-deviation of inequality $\gamma_t \equiv \frac{\Gamma_t - \Gamma}{\Gamma} = y_t^S - y_t^H$ we have the log-deviation of the Gini:

$$v_t = (1-\lambda)\frac{Y^S}{Y}\left(y_t^S - y_t\right) = \frac{\lambda(1-\lambda)\Gamma}{\lambda + (1-\lambda)\Gamma}\gamma_t,$$

which around a symmetric SS simplifies to $v_t = \lambda(1-\lambda)\gamma_t$.

A **generalized entropy** measure (with largest sensitivity to small incomes) is:

$$\Xi_t = -\lambda \ln \frac{Y_t^H}{Y_t} - (1-\lambda)\ln \frac{Y_t^S}{Y_t}$$

Subtracting the steady-state value of this same measure we obtain the deviation (note, in a uniform steady-state this measure is zero so we express this deviation in levels)

$$\begin{aligned}
\xi_t &= \Xi_t - \Xi = -\lambda \ln \frac{Y_t^H Y}{Y^H Y_t} - (1-\lambda)\ln \frac{Y_t^S Y}{Y_t Y^S} \\
&= y_t - \lambda y_t^H - (1-\lambda) y_t^S = \lambda\left(\frac{Y^H}{Y} - 1\right) y_t^H + (1-\lambda)\left(\frac{Y^S}{Y} - 1\right) y_t^S \\
&= \lambda(1-\lambda)\frac{Y^S - Y^H}{Y}\gamma_t = \lambda(1-\lambda)\frac{\Gamma - 1}{\lambda + (1-\lambda)\Gamma}\gamma_t = \frac{\Gamma - 1}{\Gamma} v_t.
\end{aligned}$$

O.B.2 Cyclical inequality: derivations

Take first the hand-to-mouth, who consume all *their* income and loglinearize the budget constraint: $c_t^H = y_t^H = w_t + n_t^H + \frac{\tau^D}{\lambda} d_t$. Substituting the wage schedule derived using the economy resource constraint, production function, and aggregate labor supply $w_t = \left(\varphi + \sigma^{-1}\right) c_t$; the profit function $d_t = -w_t$; and their labor supply, we obtain H's consumption function given in text: $c_t^H = y_t^H = \chi y_t$, with

$$\chi \equiv 1 + \varphi\left(1 - \frac{\tau^D}{\lambda}\right) \lesseqgtr 1.$$

Cyclical distributional effects make χ different from 1. The other agents, S, with income $y_t^S = w_t + n_t^S + \frac{1-\tau^D}{1-\lambda} d_t$, face an additional (relative to RANK) *income effect* of the real wage, which reduces their profits $d_t = -w_t$. Using this and their labor supply, we obtain: $c_t^S = \frac{1-\lambda\chi}{1-\lambda} y_t$, so whenever $\chi < 1$ S's income elasticity to aggregate income is *larger* than one, and vice versa.

In RANK, such distributional considerations are absent since one agent works and receives all the profits. When aggregate income goes up, labor demand goes up and the real wage increases. This drives down profits (wage=marginal cost), but because the *same* agent incurs both the labor gain and the "capital" (monopolistic rents) loss, the distribution of income between the two is neutral.

Income distribution matters under heterogeneity; to understand how, start with no fiscal redistribution, $\tau^D = 0$ and $\chi > 1$. If demand goes up and, with upward-sloping labor supply $\varphi > 0$, the wage goes up, H's income increases. Their demand increases proportionally, as they do not get hit by profits falling. Thus aggregate demand increases by *more* than initially, shifting labor demand and increasing the wage even further, and so on. In the new equilibrium, the extra demand is produced by S, whose decision to work more is optimal given the income loss from falling profits. Since the income of H goes up and down more than proportionally with aggregate income, inequality is *countercyclical*: it goes down in expansions and up in recessions.

Redistribution $\tau^D > 0$ dampens this channel, lowering χ. Through the transfer, H start internalizing the negative income effect of profits, and increase demand by less. The benchmark considered by Campbell and Mankiw's (1989) seminal paper is $\chi = 1$, which occurs when the distribution of profits is uniform $\tau^D = \lambda$ (the income effect disappears) or when labor is infinitely elastic $\varphi = 0$ (all households' consumption comoves perfectly with the wage); income inequality is then *acyclical*.

Finally, $\chi < 1$ occurs when H receive a disproportionate share of the profits $\tau^D > \lambda$. The AD expansion is now *smaller* than the initial impulse, as H recognize that this will lead to a fall in their income; while S, given the positive income effect from profits, optimally work less. As the income of H now moves less than proportionally with aggregate income, inequality is *procyclical*.

O.B.3 Income processes

Higher moments of the income process are readily calculated in my model, since individual income follows a two-state Markov chain with values Y_t^S and Y_t^H in the respective states. The analytical characterization of this process' key moments is useful both to illustrate a key dimension along which this model is a representation of complex HANK models, and for calibration and quantitative analysis. (Note that this is essentially just an analytical two-state version of the Rouwenhorst method.)

Conditional variance is found using $E_t\left(\ln Y_{t+1}^S | \ln Y_t^S\right) = s\left(Y_{t+1}\right)\ln Y_{t+1}^S + \left(1 - s\left(Y_{t+1}\right)\right)\ln Y_{t+1}^H$ as:

$$var\left(\ln Y_{t+1}^S | \ln Y_t^S\right) = s\left(Y_{t+1}\right)\left(1 - s\left(Y_{t+1}\right)\right)\left(\ln \frac{Y_{t+1}^S}{Y_{t+1}^H}\right)^2 = s\left(Y_{t+1}\right)\left(1 - s\left(Y_{t+1}\right)\right)\left(\ln \Gamma_{t+1}\right)^2. \quad \text{(O.B.1)}$$

Conditional skewness and **kurtosis** are easily calculated as:

$$\begin{aligned} skew\left(\ln Y^S_{t+1} | \ln Y^S_t\right) &= \frac{1-2s\left(Y_{t+1}\right)}{\sqrt{s\left(Y_{t+1}\right)\left(1-s\left(Y_{t+1}\right)\right)}}; \\ kurt\left(\ln Y^S_{t+1} | \ln Y^S_t\right) &= \left[s\left(Y_{t+1}\right)\left(1-s\left(Y_{t+1}\right)\right)\right]^{-1} - 3. \end{aligned} \tag{O.B.2}$$

The **first-order autocorrelation** of the income process for any of the two states $j = S, H$:

$$corr\left(\ln Y^j_{t+1}, \ln Y^j_t\right) = s + h - 1 = 1 - \frac{1-s}{\lambda}. \tag{O.B.3}$$

As standard for Bernoulli distributions there is *negative skewness* for $s > .5$ and *leptokurtosis* (positive excess kurtosis $kurt\,(.) - 3$) outside of the $\frac{1}{2} \pm \frac{1}{\sqrt{12}}$ interval, i.e. for s smaller than 0.21 or larger than 0.79. Notice that $s > 0.79$ ensures both negative skewness and leptokurtosis, with $s \geq 1 - h$ ensuring positive autocorrelation.

Of special importance to fit key micro facts on income distribution in the cross-section are the relative skewness and kurtosis of the two types. A large literature (focusing nevertheless on skewness in growth rates, not *levels*, e.g. Guvenen et al (2014)) seems to provide some support for the view that the income of an empirical proxy of S is relatively more negatively skewed and more leptokurtic. It can be easily shown, comparing (O.B.2) with the equivalent formulae for H that both properties are satisfied in the model if and only if: $s > h$. This simple two-state model features, albeit in a stylized way, some key elements of the literature pertaining to income heterogeneity and uncertainty: conditional idiosyncratic variance that can be cyclical, autocorrelated income processes with left-skewness and leptokurtosis. The combined conditions for matching the key micro facts are $s > 1 - h$, $s > h$ and both s and h larger than .79. We use this when calibrating the model in text.

The cyclicality of conditional skewness is:

$$\frac{d\,(skew)}{dY} = -\frac{s_Y}{2\left[s\,(1-s)\right]^{\frac{3}{2}}} \tag{O.B.4}$$

and is entirely determined by the cyclicality of the probability to become constrained. When the probability to become constrained $1 - s$ is increasing in recessions, $-s_Y < 0$, *risk* (in the Mankiw sense) is countercyclical: negative skewness becomes more negative in recessions, making upward income movements less likely and downward income movements more likely therein. Notice that this does not depend on the size of income inequality.

THANK thus captures analytically persistent and conditionally volatile idiosyncratic income, and also, albeit in a stylized way given the coarse two-state implicit discretization, the key feature of concomitant left-skewness and leptokurtosis that a discretization with more states matches very well. Using (O.B.2), the conditional skewness and excess kurtosis are, for the calibration used in text to match iMPCs, -1.66 and 0.77 respectively; while the quarterly autocorrelation is $(s+h-1)^{1/4} = (1-(1-s)/\lambda)^{1/4} =$ 0.819 (corresponding to the quarterly transition probability $1-s = 1-0.82^{1/4}$= $1-0.952 \simeq .04$). Given the coarse two-state discretization, it is no surprise that these moments are not perfectly aligned with the micro data.

O.C The 3-equation THANK with NKPC

This section derives the same results as in text but with the forward-looking NKPC (7).

O.C.1 The HANK Taylor Principle: Equilibrium Determinacy with Interest Rate Rules

Determinacy can be studied by standard techniques, extending the result in text (there will now be two eigenvalues). Necessary and sufficient conditions are provided i.a. in Woodford (2003) Proposition C.1. With the Taylor rule (17), the system becomes $\begin{pmatrix} E_t\pi_{t+1} & E_t c_{t+1} \end{pmatrix}' = A \begin{pmatrix} \pi_t & c_t \end{pmatrix}'$ with transition matrix:

$$A = \begin{bmatrix} \beta^{-1} & -\beta^{-1}\kappa \\ \delta^{-1}\sigma\frac{1-\lambda}{1-\lambda\chi}\left(\phi_\pi - \beta^{-1}\right) & \delta^{-1}\left(1+\sigma\frac{1-\lambda}{1-\lambda\chi}\beta^{-1}\kappa\right) \end{bmatrix}$$

with determinant $\det A = \beta^{-1}\delta^{-1}\left(1+\kappa\sigma\frac{1-\lambda}{1-\lambda\chi}\phi_\pi\right)$ and trace $\mathrm{tr}A = \beta^{-1}+\delta^{-1}\left(1+\sigma\frac{1-\lambda}{1-\lambda\chi}\beta^{-1}\kappa\right)$.

Determinacy can obtain in either of two cases. Case 2. ($\det A - \mathrm{tr}A < -1$ and $\det A + \mathrm{tr}A < -1$) can be ruled based on sign restrictions. Case 1. requires three conditions to be satisfied jointly:

$$\det A > 1;\ \det A - \mathrm{tr}A > -1;\ \det A + \mathrm{tr}A > -1$$

The third condition is always satisfied under the sign restrictions, so the necessary and sufficient conditions are:

$$\phi_\pi > 1 + \frac{(\delta-1)(1-\beta)}{\kappa\sigma\frac{1-\lambda}{1-\lambda\chi}} \tag{O.C.1}$$

together with $\phi_\pi > \max\left(\frac{\beta\delta-1}{\kappa\sigma\frac{1-\lambda}{1-\lambda\chi}},\ 1+\frac{(1-\beta)(\delta-1)}{\kappa\sigma\frac{1-\lambda}{1-\lambda\chi}}\right)$. The second term is larger than the first iff $(2\beta-1)\delta < \kappa\sigma\frac{1-\lambda}{1-\lambda\chi}+\beta$, which holds generically for most plausible parameterizations. Condition (O.C.1) thus generalizes the HANK Taylor principle to the case of forward-looking Phillips curve.

O.C.2 Ruling out the FG Puzzle

The analogous of Proposition 2 for the case with NKPC (7) is:

Proposition 9 *The analytical HANK model (with (7)) under a peg is locally determinate and solves the FG puzzle* ($\frac{\partial^2 c_t}{\partial(-i^*_{t+T})\partial T} < 0$) *if and only if:*

$$\delta + \sigma \frac{1-\lambda}{1-\lambda\chi} \frac{\kappa}{1-\beta} < 1,$$

Notice that the condition nests the one of Proposition 2 when $\beta \to 0$. Indeed, it has exactly the same interpretation with $\delta + \sigma\frac{1-\lambda}{1-\lambda\chi}\frac{\kappa}{1-\beta}$ being the "long-run" effect of news, and $\frac{\kappa}{1-\beta}$ being the slope of the long-run NKPC.

Point 1. (determinacy under a peg with NKPC) follows directly from (O.C.1): a peg is sufficient if both $\delta < \beta^{-1}$ and $1 + \frac{(1-\beta)(\delta-1)}{\kappa\sigma\frac{1-\lambda}{1-\lambda\chi}} < 0$, the latter implying $\delta < 1 - \frac{\kappa}{1-\beta}\sigma\frac{1-\lambda}{1-\lambda\chi} < \beta^{-1}$, which delivers the threshold in the Proposition.

Point 2 requires solving the model; focusing therefore on the case where the condition holds, and the model is determinate under a peg, we rewrite the model in forward (matrix) form as:

$$\begin{pmatrix} \pi_t \\ c_t \end{pmatrix} = A^{-1} \begin{pmatrix} E_t\pi_{t+1} \\ E_t c_{t+1} \end{pmatrix} - \sigma\frac{1-\lambda}{1-\lambda\chi}\begin{pmatrix} \kappa \\ 1 \end{pmatrix} i^*_t \tag{O.C.2}$$

where

$$A^{-1} = \begin{pmatrix} \beta + \kappa\sigma\frac{1-\lambda}{1-\lambda\chi} & \kappa\delta \\ \sigma\frac{1-\lambda}{1-\lambda\chi} & \delta \end{pmatrix}$$

is the inverse of matrix A defined above under a peg $\phi = 0$. To find the elasticity of $\begin{pmatrix} \pi_t & c_t \end{pmatrix}'$ to an interest rate cut at T, $-i^*_{t+T}$ we iterate forward (O.C.2) to obtain $\sigma\frac{1-\lambda}{1-\lambda\chi}\left(A^{-1}\right)^T \begin{pmatrix} \kappa \\ 1 \end{pmatrix}$. But notice that we know by point 1 that the eigenvalues of A are both outside the unit circle; it follows by standard linear algebra results that the eigenvalues of A^{-1} are both inside the unit circle and therefore $\left(A^{-1}\right)^T$ is decreasing with T. (the eigenvalues to the power of T appear in the Jordan decomposition used to compute the power of A^{-1}). This proves that the FG puzzle is eliminated.

Point 3 requires computing the equilibrium given an AR1 interest rate with persistence μ as before $E_t i^*_{t+1} = \mu i^*_t$; since we are in the determinate case, the equilibrium is unique and there is no endogenous persistence, so the persistence of endogenous variables is equal to the persistence of the exogenous process. Replacing $E_t c_{t+1} = \mu c_t$ and $E_t\pi_{t+1} = \mu\pi_t$ in (O.C.2) we therefore have:

$$\begin{pmatrix} \pi_t \\ c_t \end{pmatrix} = -\sigma\frac{1-\lambda}{1-\lambda\chi}\left(I - \mu A^{-1}\right)^{-1}\begin{pmatrix} \kappa \\ 1 \end{pmatrix} i^*_t.$$

Computing the inverse we obtain

$$\left(I-\mu A^{-1}\right)^{-1}=\frac{1}{\det}\left[\begin{array}{cc} 1-\delta\mu & \kappa\delta\mu \\ \sigma\frac{1-\lambda}{1-\lambda\chi}\mu & 1-\left(\beta+\sigma\frac{1-\lambda}{1-\lambda\chi}\kappa\right)\mu \end{array}\right],$$

where $\det \equiv \mu^2\beta\delta-\mu\left(\delta+\sigma\frac{1-\lambda}{1-\lambda\chi}\kappa+\beta\right)\mu+1$. Replacing in the previous equation, differentiating, and simplifying, the effects are:

$$\left(\begin{array}{c} \frac{\partial\pi_t}{\partial i_t^*} \\ \frac{\partial c_t}{\partial i_t^*} \end{array}\right)=-\sigma\frac{1-\lambda}{1-\lambda\chi}\frac{1}{\det}\left(\begin{array}{c} \kappa \\ 1-\mu\beta \end{array}\right)$$

Therefore, neo-Fisherian effects are ruled out iff $\det > 0$, i.e.:

$$\delta<\frac{1-\beta\mu-\sigma\frac{1-\lambda}{1-\lambda\chi}\kappa\mu}{\mu\left(1-\beta\mu\right)}.$$

But this is always satisfied under the condition in the proposition (for determinacy under a peg) $\delta < 1-\frac{\sigma\frac{1-\lambda}{1-\lambda\chi}\kappa}{1-\beta}\leq\frac{1-\beta\mu-\sigma\frac{1-\lambda}{1-\lambda\chi}\kappa\mu}{\mu(1-\beta\mu)}$ where the second inequality can be easily verified, it implies $(1-\beta\mu)(1-\beta)+\beta\sigma\kappa\mu\geq 0$.

Figure OA2 illustrates the threshold level of endogenous redistribution sufficient to deliver determinacy under a peg and thus rule out the FG puzzle, as a function of λ and for different $1-s$. Close to the TANK limit (small $1-s$), no level of redistribution delivers this (red dash); as idiosyncratic risk $1-s$ increases (blue solid), the region expands and is largest in the iid case (blue dots).

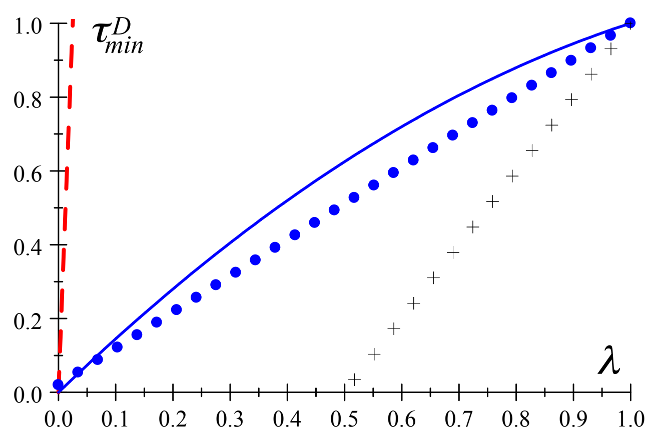

Fig. OA2: Redistribution threshold τ^D_{min} in TANK $1-s\longrightarrow 0$ (dash); 0.04 (solid); λ (dots).

Note: The crosses represent the threshold above which the IS slope is positive $\lambda\chi<1$.

O.C.3 Ruling out puzzles with Wicksellian rule and Contemporaneous PC

This Appendix provides the proof of Proposition 4 (extended to NKPC in Appendix O.C.4) and further discussion; the intuition is that, no matter how strong the AD-amplification, this rule anchors long-run expectations. Agents recognize that bygones are not bygones: adjustment will eventually take place, as inflation will *a fortiori* imply future deflation. The same intuition applies for ruling out the FG puzzle while delivering amplification, resolving the Catch-22. The essence is that under the Wicksellian rule THANK reduces, instead of one difference equation (18), to a *second-order* equation; replacing (8) rewritten with the price level $p_t - p_{t-1} = \kappa c_t$ and $i_t = \phi_p p_t$ in (16) delivers:

$$E_t p_{t+1} - \left[1 + \nu_0^{-1}\left(1 + \sigma\frac{1-\lambda}{1-\lambda\chi}\phi_p\kappa\right)\right] p_t + \nu_0^{-1} p_{t-1} = \sigma\frac{1-\lambda}{1-\lambda\chi}\kappa\nu_0^{-1} i_t^*. \tag{O.C.3}$$

Notice that the RANK model is nested here for $\lambda = 0$ (or $\chi = 1$, the Campbell-Mankiw benchmark), which would yield a simplified version of Woodford and Giannoni's analyses.

It is easy to show that (O.C.3) has a unique solution iff $\phi_p > 0$, and that the effect of an interest cut $\partial c_t / \partial\left(-i_{t+T}^*\right)$ decreases with the time T, i.e. the FG puzzle disappears. This is particularly important in HANK, for even when heterogeneity *aggravates* the puzzle, the rule restores standard logic and resolves the "Catch-22".[35]

Recall that we are interested in the case whereby $\nu_0 \geq 1$ (as the paper shows, for $\nu_0 < 1$ there there is determinacy under a peg in HANK and thus no puzzles). The model has a locally unique equilibrium (is determinate) when the above second-order equation has one root inside and one outside the unit circle. The characteristic polynomial is $J(x) = x^2 - \left[1 + (\nu_0)^{-1}\left(1 + \sigma\frac{1-\lambda}{1-\lambda\chi}\phi_p\kappa\right)\right] x + \nu_0^{-1}$ where by standard results, the roots' sum is $1 + \nu_0^{-1}\left(1 + \sigma\frac{1-\lambda}{1-\lambda\chi}\phi_p\kappa\right)$ and the product is $\nu_0^{-1} < 1$. So at least one root is inside the unit circle, and we need to rule out that both are; Since we have $J(1) = -\nu_0^{-1}\sigma\frac{1-\lambda}{1-\lambda\chi}\phi_p\kappa$ and $J(-1) = 2 + 2\nu_0^{-1} + \nu_0^{-1}\sigma\frac{1-\lambda}{1-\lambda\chi}\phi_p\kappa$, the necessary and sufficient condition for the second root to be outside the unit circle is precisely $\phi_p > 0$—coming from $J(1) < 0$ and $J(-1) > 0$. This completes the proof of Proposition 4.

To find the solution, denote the roots of the polynomial by $x_+ > 1 > x_- > 0$; the difference equation is solved by standard factorization: The roots of the characteristic polynomial are

$$\begin{aligned} x_{\pm} &= \frac{1 + \nu_0^{-1}\left(1 + \sigma\frac{1-\lambda}{1-\lambda\chi}\phi_p\kappa\right) \pm \sqrt{\left[1 + \nu_0^{-1}\left(1 + \sigma\frac{1-\lambda}{1-\lambda\chi}\phi_p\kappa\right)\right]^2 - 4\nu_0^{-1}}}{2} \\ x_+ &> 1 > x_- > 0 \end{aligned}$$

[35] A hitherto unnoticed to my knowledge corrollary is that in RANK too, the puzzle disappears under a Wicksellian rule.

Factorizing the difference equation (O.C.3):

$$\left(L^{-1}-x_{-}\right)\left(L^{-1}-x_{+}\right)p_{t-1}=\sigma\frac{1-\lambda}{1-\lambda\chi}\kappa\nu_0^{-1}i_t^*$$

we obtain:

$$\begin{aligned} p_t &= x_{-}p_{t-1}-\sigma\frac{1-\lambda}{1-\lambda\chi}\kappa\nu_0^{-1}x_{+}^{-1}\frac{1}{1-\left(x_{+}L\right)^{-1}}i_t^* \\ &= x_{-}p_{t-1}-\sigma\frac{1-\lambda}{1-\lambda\chi}\kappa\nu_0^{-1}x_{+}^{-1}\sum_{j=0}^{\infty}x_{+}^{-j}i_{t+j}^* \end{aligned}$$

Let $\Delta_{t+j}\equiv-\sigma\frac{1-\lambda}{1-\lambda\chi}\kappa\nu_0^{-1}x_{+}^{-1}i_{t+j}^*$ denote the rescaled interest rate *cut*:

$$p_t=x_{-}^{t+1}p_{-1}+\left[\sum_{j=0}^{\infty}\left(x_{+}\right)^{-j}\Delta_{t+j}+x_{-}\sum_{j=0}^{\infty}\left(x_{+}\right)^{-j}\Delta_{t-1+j}+...+x_{-}^{t-1}\sum_{j=0}^{\infty}\left(x_{+}\right)^{-j}\Delta_{1+j}+x_{-}^{t}\sum_{j=0}^{\infty}\left(x_{+}\right)^{-j}\Delta_{j}\right]$$

Normalizing initial value to zero (since $x_{-}<1$ it vanishes when t goes to infinity), the solution is made of a forward and a backward component:

$$p_t=\frac{1-\left(x_{-}x_{+}^{-1}\right)^{t+1}}{1-x_{-}x_{+}^{-1}}\sum_{j=0}^{\infty}\left(x_{+}^{-1}\right)^{j}\Delta_{t+j}+\sum_{k=0}^{t-1}x_{-}^{1+k}\frac{1-\left(x_{-}x_{+}^{-1}\right)^{t-k}}{1-x_{-}x_{+}^{-1}}\Delta_{t-1-k}$$

Lagging it once and taking the first difference we obtain the solution for inflation:

$$\begin{aligned} \pi_t &= \frac{1-\left(x_{-}x_{+}^{-1}\right)^{t+1}}{1-x_{-}x_{+}^{-1}}\sum_{j=0}^{\infty}\left(x_{+}^{-1}\right)^{j}\Delta_{t+j}-\frac{1-\left(x_{-}x_{+}^{-1}\right)^{t}}{1-x_{-}x_{+}^{-1}}\sum_{j=0}^{\infty}\left(x_{+}^{-1}\right)^{j}\Delta_{t-1+j} \\ &\quad+\sum_{k=0}^{t-1}x_{-}^{1+k}\frac{1-\left(x_{-}x_{+}^{-1}\right)^{t-k}}{1-x_{-}x_{+}^{-1}}\Delta_{t-1-k}-\sum_{k=0}^{t-2}x_{-}^{1+k}\frac{1-\left(x_{-}x_{+}^{-1}\right)^{t-1-k}}{1-x_{-}x_{+}^{-1}}\Delta_{t-2-k} \\ &= A\left(t\right)\sum_{j=0}^{\infty}\left(x_{+}^{-1}\right)^{j}\Delta_{t+j}+\Psi_{t-1}. \end{aligned}$$

where $A\left(t\right)\equiv\frac{1-\left(x_{+}^{-1}\right)+\left(x_{-}\right)^{t}\left(x_{+}^{-1}\right)^{t+1}-\left(x_{-}x_{+}^{-1}\right)^{t+1}}{1-x_{-}x_{+}^{-1}}$ (if we put ourselves at time 0 this simply becomes $A\left(0\right)=\sigma\frac{1-\lambda}{1-\lambda\chi}\nu_0^{-1}$), while in Ψ_{t-1} we grouped all terms that consist of lags of Δ_t (Δ_{t-1} and earlier) which are predetermined at time t and will not be used in any of the derivations of interest here—where we consider shocks occurring at t or thereafter. This delivers, for consumption:

$$c_t=-A\left(t\right)E_t\sum_{j=0}^{\infty}\left(x_{+}^{-1}\right)^{j+1}i_{t+j}^*+\Psi_{t-1} \tag{O.C.4}$$

where Ψ_{t-1} is a weighted sum of past realizations of the shock and $A(t) > 0$ is a function only of calendar date; both Ψ_{t-1} and $A(t)$ are irrelevant for our purpose because they are invariant to current and future shocks.

The effect of a one-time interest rate cut at $t+T$ is now:

$$\frac{\partial c_t}{\partial\left(-i^*_{t+T}\right)} = A(t)\left(x_+^{-1}\right)^{T+1}$$

which, since $A(.) > 0$ and $x_+ > 1$, is a decreasing function of T: *the FG puzzle disappears*.[36] Notice that the Wicksellian rule *also* cures the FG puzzle in the (nested) RANK model (this follows immediately by replacing $\lambda = 0$ or $\chi = 1$ above).

O.C.4 Determinacy with Wicksellian rule and NKPC

Rewrite the system made of (13), (7) and the definition of inflation as (ignoring shocks):

$$\begin{aligned} c_t &= \delta E_t c_{t+1} - \sigma\frac{1-\lambda}{1-\lambda\chi}\phi_p p_t + \sigma\frac{1-\lambda}{1-\lambda\chi}E_t\pi_{t+1} \\ \pi_t &= \beta E_t\pi_{t+1} + \kappa c_t \\ p_t &= \pi_t + p_{t-1} \end{aligned}$$

Substituting and writing in canonical matrix form $\begin{pmatrix} E_t c_{t+1} & E_t\pi_{t+1} & p_t \end{pmatrix}' = A\begin{pmatrix} c_t & \pi_t & p_{t-1} \end{pmatrix}'$ with transition matrix A given by

$$A = \begin{pmatrix} \delta^{-1}\left(1+\beta^{-1}\sigma\frac{1-\lambda}{1-\lambda\chi}\kappa\right) & \delta^{-1}\sigma\frac{1-\lambda}{1-\lambda\chi}\left(\phi_p - \beta^{-1}\right) & \delta^{-1}\sigma\frac{1-\lambda}{1-\lambda\chi}\phi_p \\ -\beta^{-1}\kappa & \beta^{-1} & 0 \\ 0 & 1 & 1 \end{pmatrix}.$$

We can apply Proposition C.2 in Woodford (2003, Appendix C): determinacy requires two roots outside the unit circle and one inside. The characteristic equation of matrix A is:

$$J(x) = x^3 + A_2x^2 + A_1x + A_0 = 0$$

[36] Likewise for neo-Fisherian effects: take an AR(1) process for i^*_t with persistence μ as before; the solution is now both 1. uniquely determined (by virtue of determinacy proved above) and 2. in line with standard logic—an increase in interest rates leads to a fall in consumption and deflation in the short run: $\frac{\partial c_t}{\partial i^*_t} = -A(t)\frac{1}{x_+-\mu}$,which is negative as $A(.) > 0$ and $x_+ > 1 > \mu$. Notice that in the long-run, i.e. if there is a permanent change in interest rates, the economy moves to a new steady-state and the uncontroversial. long-run Fisher effect applies as usual.

with coefficients:

$$
\begin{aligned}
A_2 &= -\frac{1}{\beta} - \frac{1}{\delta}\left(\frac{\sigma\kappa}{\beta}\frac{1-\lambda}{1-\lambda\chi}+1\right) - 1 < 0 \\
A_1 &= \frac{1}{\beta} + \frac{1}{\delta}\left[\frac{\sigma\kappa}{\beta}\frac{1-\lambda}{1-\lambda\chi}\left(1+\phi_p\right)+1+\frac{1}{\beta}\right] > 0 \\
A_0 &= -\frac{1}{\beta\delta}
\end{aligned}
$$

To check the determinacy conditions, we first calculate:

$$
\begin{aligned}
J(1) &= 1 + A_2 + A_1 + A_0 = \frac{1}{\delta}\frac{\sigma\kappa}{\beta}\frac{1-\lambda}{1-\lambda\chi}\phi_p > 0 \\
J(-1) &= -1 + A_2 - A_1 + A_0 \\
&= -2 - \frac{2}{\beta} - \frac{1}{\delta}\left[2\frac{\sigma\frac{1-\lambda}{1-\lambda\chi}\kappa}{\beta} + \frac{\sigma\frac{1-\lambda}{1-\lambda\chi}\kappa}{\beta}\phi_p + 2 + \frac{2}{\beta}\right] < 0
\end{aligned}
$$

Since $J(1) > 0$ and $J(-1) < 0$ we are either in case Case II or Case III in Woodford Proposition C.2;

Case III in Woodford implies that $\phi_p > 0$ is sufficient for determinacy if the additional condition is satisfied:

$$
A_2 < -3 \rightarrow \delta < \frac{\sigma\frac{1-\lambda}{1-\lambda\chi}\kappa + \beta}{2\beta - 1}. \tag{O.C.5}
$$

This is a fortiori satisfied in RANK (and delivers determinacy there), but not here with $\delta > 1$. Therefore, we also need to check Case II in Woodford and to that end we need to check the additional requirement (C.15) therein:

$$
A_0^2 - A_0 A_2 + A_1 - 1 > 0,
$$

which replacing the expressions for the A_is delivers:

$$
\phi_p > \frac{(1-\beta)(\delta-1) + \sigma\frac{1-\lambda}{1-\lambda\chi}\kappa}{\sigma\frac{1-\lambda}{1-\lambda\chi}\kappa\delta\beta}(1-\delta\beta)
$$

Since the ratio is positive, this requirement is only stronger than the already assumed $\phi_p > 0$ when

$$
\delta < \beta^{-1}; \tag{O.C.6}
$$

It can be easily checked that the δ threshold O.C.6 is always smaller than the threshold O.C.5; therefore, whenever $\delta < \beta^{-1}$, Case III applies and $\phi_p > 0$ is sufficient for determinacy. While when O.C.5 fails (for large enough δ), Case II applies and $\phi_p > 0$ is still sufficient for determinacy.

O.D Thank with Liquidity, derivations

O.D.1 Proof of Proposition 8

The solution of the asset-accumulation equation implies the following recursions for the responses of assets to income shocks:

$$
\begin{aligned}
t &\leq T-1\text{: } \frac{db_{t+1}}{d\hat{y}_T} = x_b \frac{db_t}{d\hat{y}_T} + \frac{1-\lambda}{s} \left(\beta x_b\right)^{T-t} \left(\beta x_b - \delta\right) \\
t &= T: \frac{db_{t+1}}{d\hat{y}_T} = x_b \frac{db_t}{d\hat{y}_T} + \frac{1-\lambda}{s} \beta x_b \\
t &> T: \frac{db_{t+1}}{d\hat{y}_T} = x_b \frac{db_t}{d\hat{y}_T}
\end{aligned}
$$

The solutions of these equations are (setting initial debt equal to steady-state without loss of generality):

$$
\begin{aligned}
t &\leq T-1\text{:} \frac{db_{t+1}}{d\hat{y}_T} = \left(\beta x_b\right)^{T-t} \frac{1-\lambda\chi}{s} \left(\beta x_b - \delta\right) \frac{1-\left(x_b \beta x_b\right)^{t+1}}{1-x_b \beta x_b} \\
t &= T: \frac{db_{T+1}}{d\hat{y}_T} = x_b \beta x_b \frac{1-\lambda\chi}{s} \left(\beta x_b - \delta\right) \frac{1-\left(x_b \beta x_b\right)^{T}}{1-x_b \beta x_b} + \frac{1-\lambda\chi}{s} \beta x_b \\
t &\geq T+1: \frac{db_{t+1}}{d\hat{y}_T} = x_b^{t-T} \frac{db_{T+1}}{d\hat{y}_T}
\end{aligned}
$$

Taking derivatives of the consumption function 50, we have:

$$
\begin{aligned}
t &\leq T-1\text{: } \frac{dc_t}{d\hat{y}_T} = \beta^{-1}\left(1-\beta x_b\right) \frac{db_t}{d\hat{y}_T} + \frac{1-\lambda\chi}{s} \left(\beta x_b\right)^{T-t} \left(\delta - \beta x_b\right) \\
t &= T: \frac{dc_t}{d\hat{y}_T} = 1 + \beta^{-1}\left(1-\beta x_b\right) \frac{db_t}{d\hat{y}_T} - \frac{1-\lambda\chi}{s} \beta x_b \\
t &> T: \frac{dc_t}{d\hat{y}_T} = \beta^{-1}\left(1-\beta x_b\right) \frac{db_t}{d\hat{y}_T}
\end{aligned}
$$

Replacing the solution for assets:

$$
\begin{aligned}
t &\leq T-1: \frac{dc_t}{d\hat{y}_T} = \beta^{-1}\left(1-\beta x_b\right) \left(\beta x_b\right)^{T-t+1} \frac{1-\lambda\chi}{s} \left(\beta x_b - \delta\right) \frac{1-\left(x_b \beta x_b\right)^{t}}{1-x_b \beta x_b} + \frac{1-\lambda\chi}{s} \left(\beta x_b\right)^{T-t} \left(\delta - \beta x_b\right) \\
t &= T: \frac{dc_t}{d\hat{y}_T} = 1 + \beta^{-1}\left(1-\beta x_b\right) \beta x_b \frac{1-\lambda\chi}{s} \left(\beta x_b - \delta\right) \frac{1-\left(x_b \beta x_b\right)^{T}}{1-x_b \beta x_b} - \frac{1-\lambda\chi}{s} \beta x_b \\
t &\geq T+1: \frac{dc_t}{d\hat{y}_T} = \beta^{-1}\left(1-\beta x_b\right) \frac{db_t}{d\hat{y}_T} = \beta^{-1}\left(1-\beta x_b\right) x_b^{t-T-1} \frac{db_{T+1}}{d\hat{y}_T} \\
&= \beta^{-1}\left(1-\beta x_b\right) x_b^{t-T-1} \left(x_b \beta x_b \frac{1-\lambda\chi}{s} \left(\beta x_b - \delta\right) \frac{1-\left(x_b \beta x_b\right)^{T}}{1-x_b \beta x_b} + \frac{1-\lambda\chi}{s} \beta x_b \right)
\end{aligned}
$$

Rewriting and simplifying, we obtain the expressions in Proposition 8. Notice that, as argued by

Auclert et al, the present discounted sum of the iMPCs needs to be 1 (the increase in income is consumed entirely, sooner or later). To prove that the iMPCs in THANK derived here satisfy this property, replace the respective solution into the sum:

$$\sum_{t=0}^{T-1} \beta^{t-T} \frac{dc_t}{d\hat{y}_T} + \frac{dc_T}{d\hat{y}_T} + \sum_{t=T+1}^{\infty} \beta^{t-T} \frac{dc_t}{d\hat{y}_T}$$

obtaining

$$\begin{aligned} & \frac{1-\lambda\chi}{s} \frac{\delta - \beta x_b}{1-\beta x_b^2} \beta^T x_b \left[1 - \left(\beta x_b^2\right)^T\right] \\ & +1 - \frac{1-\lambda\chi}{s} \beta x_b - (\delta - \beta x_b)\, x_b \frac{1-\lambda\chi}{s} (1-\beta x_b) \frac{1-(x_b \beta x_b)^T}{1 - x_b \beta x_b} \\ & + \frac{1-\lambda\chi}{s} \frac{\beta x_b}{1-\beta x_b^2} \left(1 - x_b \delta + x_b (\delta - \beta x_b) \left(\beta x_b^2\right)^T\right) \\ = \; & 1 \end{aligned}$$

The **calibration** in text following Auclert et al concerns two iMPCs, $\frac{dc_0}{d\hat{y}_0} = 1 - \frac{1-\lambda\chi}{s} \beta x_b$ and $\frac{dc_1}{d\hat{y}_0} = \frac{1-\lambda\chi}{s} (1-\beta x_b)\, x_b$.

Part (ii) of the Proposition concerns the dependence on χ (and δ Euler-compounding), keeping fixed the time-0 contemporaneous MPC $\frac{dc_0}{d\hat{y}_0}$; denote this by:

$$m_{00} \equiv \frac{dc_0}{d\hat{y}_0} = 1 - \frac{1-\lambda\chi}{s} \beta x_b$$

Replacing in the Proposition and rewriting the iMPCs, taking the derivative with respect to the cyclicality of inequality χ we obtain:

$$\frac{\partial \frac{dc_t}{d\hat{y}_T}}{\partial \chi}|_{\overline{m_{00}}} = \frac{\partial}{\partial \chi} \begin{cases} \frac{1-m_{00}}{\beta x_b} \frac{\delta - \beta x_b}{1-\beta x_b^2} (\beta x_b)^{T-t} \left(1 - x_b + x_b (1-\beta x_b) \left(\beta x_b^2\right)^t\right), & \text{if} \quad t \leq T-1; \\ 1 - \frac{1-m_{00}}{\beta x_b} \beta x_b - (\delta - \beta x_b) \frac{1-m_{00}}{\beta} (1-\beta x_b) \frac{1-\left(\beta x_b^2\right)^T}{1-\beta x_b^2}, & \text{if} \quad t = T; \\ \frac{1-m_{00}}{\beta x_b} \frac{1-\beta x_b}{1-\beta x_b^2} x_b^{t-T} \left(1 - x_b \delta + x_b (\delta - \beta x_b) \left(\beta x_b^2\right)^T\right), & \text{if} \quad t \geq T+1. \end{cases}$$

It follows directly that "anticipation iMPCs" ($t < T$) are increasing in χ (using $\frac{\partial \delta}{\partial \chi} = (1-s) \frac{1-\lambda}{(1-\lambda\chi)^2} > 0$); iMPCs out of past income ($t > T$) are decreasing in χ (the derivative is proportional to $-x_b \left(1 - \left(\beta x_b^2\right)^T\right) \frac{\partial \delta}{\partial \chi} < 0$), and decrease the contemporaneous MPC at given T.

O.D.2 Determinacy with a nominal debt quantity rule

In this Appendix, I prove Proposition (6) for the case of static Phillips curve. As we shall see, the algebra and intuition are both very similar to the proof of determinacy under a Wicksellian rule—and so is the generalization to forward-looking Phillips curve, which I ignore for brevity.

We start from the loglinearized government budget constraint is with positive steady-state liquidity B_Y, outlined in (54) in main Appendix C.5. Next, notice that one important dimension of Hagedorn's policy is that taxes adjust automatically to ensure that the government budget constraint is indeed a constraint for any price level (thus ruling out fiscal-theory equilibria); that is, t_t is endogenous and determined by (54) residually

The aggregate Euler equation for the model with liquidity and with steady-state inequality and non-zero steady-state debt is derived in main Appendix C.5 as equation (56), reproduced here for convenience:

$$
\begin{aligned}
c_t = {} & \delta E_t c_{t+1} - \sigma \frac{1-\lambda}{1-\lambda\chi}\left(i_t - E_t \pi_{t+1}\right) \\
& - \frac{\lambda}{1-\lambda\chi} t_t - \frac{1-\lambda-s}{1-\lambda\chi} E_t t_{t+1} + \frac{1-\lambda-s}{1-\lambda\chi}\frac{1-s}{\lambda} R b_{t+1} + \frac{\lambda}{1-\lambda\chi}\frac{1-s}{\lambda} R b_t \\
& + \frac{1-\lambda-s}{1-\lambda\chi}\frac{1-s}{\lambda} R B_Y \left(i_t - E_t \pi_{t+1}\right) + \frac{\lambda}{1-\lambda\chi}\frac{1-s}{\lambda} R B_Y \left(i_{t-1} - \pi_t\right)
\end{aligned}
$$

When is the equilibrium determinate under the debt-quantity rule fixing the nominal amount of debt proposed by Hagedorn? Replace the rule $b^N_{t+1} = 0 \rightarrow b_{t+1} = b^N_{t+1} - p_t = -p_t$, taxes from (54), the assumed exogenous interest rates $i_t = i^*_t$, and the static Phillips curve $c_t = \kappa^{-1}\left(p_t - p_{t-1}\right)$ to obtain, rearranging (take log utility without loss of generality):

$$
\begin{aligned}
& \left[\delta\kappa^{-1} + \frac{1-\lambda}{1-\lambda\chi} - \frac{\lambda\left(\frac{1-s}{\lambda}-1\right)^2}{1-\lambda\chi} R B_Y - \frac{\lambda\left(\frac{1-s}{\lambda}-1\right)}{1-\lambda\chi}\right] E_t p_{t+1} \\
& - \left\{ (1+\delta)\kappa^{-1} + \frac{1-\lambda}{1-\lambda\chi} - \frac{\lambda\left(\frac{1-s}{\lambda}-1\right)^2}{1-\lambda\chi} R B_Y + \frac{\lambda\left[1+\left(\frac{1-s}{\lambda}-1\right)^2 R\right]}{1-\lambda\chi} + \frac{\lambda\left(\frac{1-s}{\lambda}-1\right)}{1-\lambda\chi} R B_Y \right\} p_t \\
& + \left[\kappa^{-1} - \frac{\lambda\left(\frac{1-s}{\lambda}-1\right)}{1-\lambda\chi} R + \frac{\lambda\left(\frac{1-s}{\lambda}-1\right)}{1-\lambda\chi} R B_Y\right] p_{t-1} = \\
& \left(\frac{1-\lambda}{1-\lambda\chi} - \frac{\lambda\left(\frac{1-s}{\lambda}-1\right)^2}{1-\lambda\chi} R B_Y\right) i^*_t - \frac{\lambda\left(\frac{1-s}{\lambda}-1\right)}{1-\lambda\chi} R B_Y i^*_{t-1}
\end{aligned}
$$

This is a second-order difference equation with characteristic polynomial

$$J(x) = A_2 x^2 - A_1 x + A_0,$$

where $A_2,\ A_1,\ A_0$ are the coefficients on $E_t p_{t+1},\ p_t,\ p_{t-1}$. By the same standard results invoked before, the necessary and sufficient condition for determinacy is $J(1)\,J(-1) < 0$, where

$$\begin{aligned}
J(1) &= -\frac{1-s}{1-\lambda\chi}\left[1 + R\left(\frac{1-s}{\lambda} - 1\right)\right] \\
J(-1) &= 2\kappa^{-1}(1+\delta) + 2\frac{1-\lambda}{1-\lambda\chi} + 2\frac{\lambda}{1-\lambda\chi}\left(\frac{1-s}{\lambda} - 1\right) R B_Y \left(2 - \frac{1-s}{\lambda}\right) \\
&\quad + \frac{\lambda}{1-\lambda\chi}\left(2 - \frac{1-s}{\lambda}\right)\left[1 + \left(1 - \frac{1-s}{\lambda}\right) R\right]
\end{aligned}$$

First, we have $J(1) < 0$ if:

$$1 + R\left(\frac{1-s}{\lambda} - 1\right) > 0 \rightarrow R < \frac{1}{1 - \frac{1-s}{\lambda}},$$

which is always satisfied (see the restrictions needed for the steady-state debt demand). Next, note that $J(-1) > 0$ holds in e.g. the most extreme case of pure oscillation $s = 0$, as $2\kappa^{-1}(1+\delta) + 2\frac{1-\lambda}{1-\lambda\chi} > 0$. In the more general case, with $1 - s < \lambda$, the third term in $J(-1)$ can in principle become negative when $B_Y > 0$ but we can in fact prove that the sufficient condition for its being positive (sum of third and fourth term is always positive): $2\left(\frac{1-s}{\lambda} - 1\right) R B_Y + 1 + \left(1 - \frac{1-s}{\lambda}\right) R > 0$ implies (using $\left(1 - \frac{1-s}{\lambda}\right) R < 1$)

$$B_Y < \frac{\frac{1}{\left(1-\frac{1-s}{\lambda}\right)R} + 1}{2} > 1$$

Replacing the equilibrium expression for debt, this implies

$$-\left(\frac{\frac{1-\lambda}{1-s}\left(\frac{1}{\beta R} - 1\right)}{1 + \frac{1-\lambda}{1-s}\left(\frac{1}{\beta R} - 1\right)} + \frac{1}{1 + (1-\lambda)(\Gamma - 1)}\right) < \frac{1 + \left(\left(\frac{1-s}{\lambda} - 1\right) R\right)^2}{2\left(1 - \frac{1-s}{\lambda}\right) R},$$

which is always satisfied since the LHS is negative and the RHS positive. This proves the Proposition in the general case with $B_Y \geq 0$. In the special case, zero-liquidity limit $B_Y = 0$ the determinacy conditions are much simpler with $J(-1) > 0$ following immediately and $J(1) < 0$ requiring:

$$\frac{1-s}{\lambda} > 1 - \beta.$$

The proof that the debt quantity rule eliminates the FG puzzle and expansionary interest rate increases is entirely analogous to the one for a Wicksellian rule outlined in Appendix O.C.3 above, with the appropriate change of notation (implying different roots x_+ and x_-). Likewise, the extension to forward-looking Phillips curve is also analogous.

Indeed, notice that in the iid idiosyncratic risk case $1-s=\lambda$, the key difference equation becomes (using the same notation $\nu_0 = \delta + \kappa\frac{1-\lambda}{1-\lambda\chi}$):

$$E_t p_{t+1} - \left[1+\nu_0^{-1}\left(1+\lambda\frac{\kappa}{1-\lambda\chi}\right)\right] p_t + \nu_0^{-1} p_{t-1} = \frac{1-\lambda}{1-\lambda\chi}\kappa\nu_0^{-1} i_t^*.$$

This has *exactly* the same form as the equation under Wicksellian rule, but with λ instead of $\sigma(1-\lambda)\phi_p$ inside the bracket of the coefficient on p_t; here, the real demand for liquidity is elastic to the price because of risk, combined with a fixed nominal quantity; under the Wicksellian rule, because of the direct effect $\sigma(1-\lambda)$ of interest rates (and their dependency on p via ϕ_p) on the saving decision of S.

Considering instead the **extension whereby the debt rule responds to the price level** $b_{t+1}^N = \phi_b p_t \rightarrow b_{t+1} = (\phi_b - 1)\, p_t$ and replacing in the Euler equation above doing the same (tedious) manipulations, we obtain the determinacy-relevant characteristic polynomial objects:

$$J(1) = -(1-\phi_b)\frac{\lambda}{1-\lambda\chi}\frac{1-s}{\lambda}\left[1+\left(\frac{1-s}{\lambda}-1\right)R\right]$$

$$\begin{aligned} J(-1) = {} & 2\kappa^{-1}(1+\delta) + 2\frac{1-\lambda}{1-\lambda\chi} + 2\frac{\lambda}{1-\lambda\chi}\left(2-\frac{1-s}{\lambda}\right)\left(\frac{1-s}{\lambda}-1\right)RB_Y \\ & + (1-\phi_b)\frac{\lambda}{1-\lambda\chi}\left(2-\frac{1-s}{\lambda}\right)\left[1+\left(1-\frac{1-s}{\lambda}\right)R\right] \end{aligned}$$

These satisfy the necessary and sufficient condition for determinacy if $\phi_b < 1$ so that $J(1) < 0$ and $J(-1) > 0$.